Multiple Kronecker Covering Graphs

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Abstract

A graph may be the Kronecker cover in more than one way. In this note we explore this phenomenon and apply it to show that the minimal common cover of two graphs need not be unique.

We provide examples of graphs of the same size with non-unique common double covers and of isovalent graphs of the same size with non-unique minimal regular common covers.

1 Introduction

A graph $\tilde{G}$ is said to be a covering graph over a graph G if there exists a surjective homomorphism (called a covering) $f\colon \tilde{G} \to G$ such that for every vertex v of $\tilde{G}$ the set of edges incident with v is mapped bijectively onto the set of edges incident with $f(v)$. A covering f is k-fold if the preimage of every vertex of G consists of k vertices. It is regular if G is the quotient graph of a group acting freely on $\tilde{G}$. (To avoid confusion we will call regular graphs isovalent in this note.)

To simplify the description of large graphs, the concept of *voltage graphs* and *covering graphs* is generally used, see for example [2] or [11].

In 1982 F.T. Leighton proved in [6] that any two graphs with a common universal cover have a common finite cover. It is not hard to see that any two graphs with a common cover have a unique maximal common cover: the universal cover. In this note we show that the result does not extend in the opposite direction. There are graphs with a common cover whose minimal common double cover is not unique. We also give examples of isovalent graphs of the same size that have non-unique minimal regular common covers.

In the first construction we use several properties of the well-known Kronecker double cover, which plays an important role in combinatorial geometries as the neighborhood geometry of a graph, see [5, 9].

In the second we exploit properties of Cayley graphs and products of graphs.

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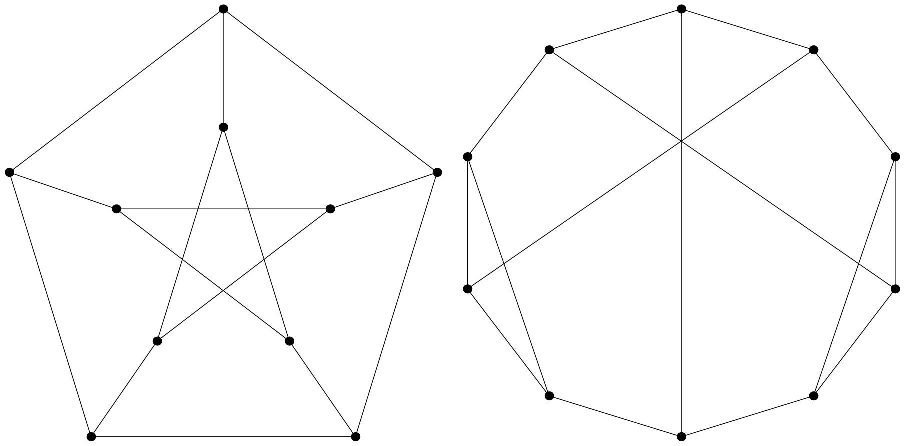

Figure 1: The Desargues graph $G(10,3)$ is a Kronecker cover of the Petersen graph $G(5,2)$ and of the graph X.

2 Graphs are not determined by their Kronecker covers

The *Kronecker cover*, also known as the *canonical double cover*, of G is the $\mathbb{Z}_2$-covering graph over G with voltages 1 on all edges (note that in this case the direction of edges is irrelevant). We will denote the Kronecker cover of G by $KC(G)$. Alternatively, $KC(G)$ can be defined as the tensor product, also known as the categorical product, see [7], of G and K_2. See [3] for more about graph products.

It is easy to see the following properties of $KC(G)$.

Proposition 1. *Kronecker covers of graphs are bipartite. If G is bipartite, then $KC(G)$ consists of two copies of G. If G is connected and non-bipartite then $KC(G)$ is connected.*

Proof. By definition, the vertex set $V(KC(G))$ is a union of two sets $V(G)$ and the edges of $KC(G)$ connect only vertices in different copies of $V(G)$. $\square$

It follows from the theory of tensor products developed by Imrich et al. that $KC(G) = KC(G')$ does not necessarily imply $G = G'$. Recently, Imrich et al. have determined all possibilities for a hypercube Q_n to be a Kronecker cover, see [1] for finite hypercubes and [4] for the weak infinite hypercube.

Here we open the problem for all simple graphs.

Problem 1. *Given a connected, simple graph K, determine all simple graphs G such that $K = KC(G)$.*

Clearly, not all graphs can be Kronecker covers. Here is a simple criterion for a graph K to be Kronecker cover. Let K be bipartite with bipartition (V_1, V_2) and let $\pi \in \operatorname{Aut} K$ be a fixed-point free involution such that π interchanges the bipartition: $\pi(V_1) = V_2$. Furthermore, we require that for any vertex v of K vertices v and $\pi(v)$ are non-adjacent. Such an automorphism is called a *(combinatorial) polarity.* Note that Kronecker covers have attracted the attention of geometers under the name of *neighborhood geometries* of graphs, see [5] and [9]. Actually, in the paper by Van Maldeghem [9] our first three propositions have been considered in the bi-slim case. In our language this means they were studied for connected trivalent (or cubic) graphs.

Proposition 2. *Let K be a connected graph. Then K is a Kronecker cover of some graph G if and only if K is bipartite and there exists a polarity $\pi \in \operatorname{Aut} K$.*

Let $\Pi K \subset \operatorname{Aut} K$ denote the set of all polarities of K. Clearly if π is a polarity and α an arbitrary automorphism, then $\pi^\alpha = \alpha\pi\alpha^{-1}$ is also a polarity, because every automorphism either fixes or interchanges the bipartition. Let $\pi^{\operatorname{Aut} K}$ denote the class $\pi^{\operatorname{Aut} K} = \{\pi^\alpha | \alpha \in \operatorname{Aut} K\}$. If we define an equivalence relation $\cong$ in ΠK so that π is equivalent to π' if and only if there exists an $\alpha \in \operatorname{Aut} K$ such that $\pi' = \pi^\alpha$, then the equivalence classes are exactly of the form $\pi^{\operatorname{Aut} K}$.

Proposition 3. *Let K be a connected graph. Then the number of simple graphs for which K is a Kronecker cover is no more than the number $|\Pi_K/{\cong}|$ of equivalence classes under $\cong$.*

Let us consider the case presented in Figure 1. The Desargues graph $G(10, 3)$ can be represented as a Kronecker covering graph in two distinct ways. Let us label the vertices of Petersen graph $G(5, 2)$ in such a way that the vertices in the outer pentagon are labeled 1,2,3,4,5 and the vertices in the inner pentagram 6,7,8,9,10 with 1 being adjacent to 6, 2 to 7, etc. The labeling of the vertices of the bipartite $G(10, 3)$ is chosen in such a way that the vertex i of Petersen lifts to a black vertex i and a white vertex i'. Each pair i and i' of vertices is antipodal in $G(10, 3)$. In order to specify the second quotient, the graph X, we have to define a new polarity π of $G(10, 3)$ that tells which white vertex $\pi(i)$ projects onto the vertex i of X. We do this with the aid of an involution α if $G(5, 2)$ defined by $\alpha = (1, 8)(2, 10)(3, 5),(4)(6)(7)(9)$ with four fixed points by setting $\pi(i) = \alpha(i)'$ and $\pi(i') = \alpha(i)$. If we now identify $\pi(i')$ with i we obtain a covering projection of $G(10, 3)$ onto the graph X labelled in the following order along the Hamilton cycle of the graph on the right side of Figure 1: $\{10, 1, 3, 4, 9, 6, 8, 5, 2, 7\}$ where $10 - 1 - 3$ and $8 - 5 - 2$ are the two triangles. More generally we have the following proposition.

Proposition 4. *Let α be an involution of a graph G that does not interchange the endpoints of an edge, then $\pi(i) = \alpha(i)'$ and $\pi(i') = \alpha(i)$ is a polarity of $KC(G)$.*

3 Least common covers are not unique

It is well-known that if two graphs share a common cover then they have the same universal cover, that is the largest possible connected cover of the two graphs.

Here we show that the converse problem, namely finding the least common cover may have more than one solution. Let G and H be disjoint connected graphs. Let $G \smile H$ be a graph composed from G and H adding an edge that connects some vertex of G to some other vertex of H. This operation depends

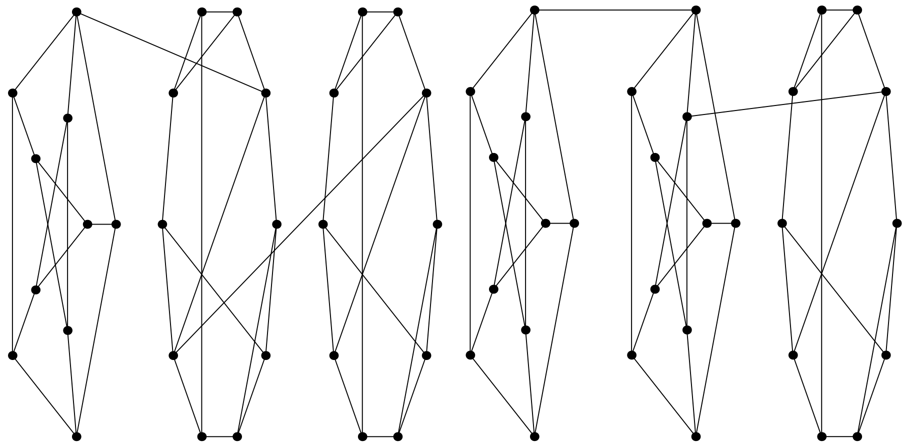

Figure 2: The graphs H_1 and H_2 have more than one common connected double cover.

on the choice of vertices, which we indicate by reference to the figure. Let G^* and H^* be any two double covers of G and H respectively. Then there is a unique way to extend this to a Kronecker double cover of $G \smile H$ that we shall denote by $G^* \asymp H^*$.

Let us take the two familiar graphs $G(5,2)$ and X. Now form two graphs $H_1 = G(5,2) \smile X \smile X$ and $H_2 = G(5,2) \smile G(5,2) \smile X$; see Figure 2. Let $G_0 = G(10,3) \asymp G(10,3) \asymp G(10,3), G_1 = G(10,3) \asymp G(10,3) \asymp 2X$ and $G_2 = 2G(5,2) \asymp G(10,3) \asymp G(10,3)$.

It is not hard to see that G_0, G_1, and G_2 cover H_1 and H_2. Clearly G_0 and G_1 are nonisomorphic because G_1 has two bridges, but not G_0. (We also used the computer system VEGA (see [10]) toe check that G_0 and G_1 are nonisomorphic double covers of H_1 and H_2.) We have thus shown:

Theorem 5. *There exist connected graphs H_1 and H_2 of the same size with a common universal cover and non-unique common double covers.*

One of the referees remarked that it is easy to prove a weaker version of this theorem by providing examples of connected graphs with the same universal cover and common nonisomorphic minimal covers. His examples, which we will discuss in the sequel, involve infinite families of isovalent graphs of different sizes. In fact, they are Cartesian products of cycles of prime lengths. To describe them we let C_n denote a cycle of length n and $\Box$ the Cartesian product of graphs, see [3] for the definition of the Cartesian product.

Let $H_1 = C_3 \Box C_4, H_2 = C_5 \Box C_7$. Then $G_1 = C_{15} \Box C_{28}, G_2 = C_{21} \Box C_{20}$ are both minimal common covers of H_1 and H_2. Note that they are also minimal common covers of $H_1' = C_3 \Box C_5, H_2' = C_4 \Box C_7$ and $H_1'' = C_3 \Box C_7, H_2'' = C_4 \Box C_7$. The same conclusion holds if we replace numbers $3, 4, 5, 7$ by any set of pairwise relatively prime integers m, n, p, q. The minimality of the common cover follows from the fact that the least common multiple of these four integers is their product.

This referee also asked for examples of isovalent graphs of the same size with common non-isomorphic minimal covers. While it is easy to see that

$C_p \Box C_q \Box C_{rs}$ and $C_r \Box C_s \Box C_{pq}$ have the same size and several common covers, for example $C_{pr} \Box C_{qs} \Box C_{pqrs}$, $C_{pr} \Box C_{pq} \Box C_{rs}$, $C_{ps} \Box C_{qr} \Box C_{pqrs}$, $C_{ps} \Box C_{pq} \Box C_{rs}$, $C_{pq} \Box C_{qr} \Box C_{rs}$, $C_{pq} \Box C_{qs} \Box C_{rs}$, only one of them, namely $C_{pr} \Box C_{pq} \Box C_{rs}$, will be minimal if $p < q < r < s$ are odd primes.

On the other hand, these graphs have common covers $C_{pr} \Box C_{qs} \Box C_{pqrs}$ and $C_{ps} \Box C_{qr} \Box C_{pqrs}$ that are non-isomorphic, have the same number of vertices, but they are not minimal.

The reason is that the factor C_{pqrs} can be mapped onto any of the factors $C_p, C_q, C_r, C_s, C_{pq}$ or C_{rs} of the covered graphs. It would be desirable to be able to change the examples such that is not possible any more. There are at least two possibilities to achieve this, for example with the direct product of graphs or the strong one. (See [3] for the definition of the direct and the strong product.)

We believe that our examples are minimal common covers, but we can only show that they are minimal regular common covers. We state our claim with respect to the strong product, which we denote by $\boxtimes$.

Theorem 6. *Let $p < q < r < s$ be odd primes. Then $(C_{pr} \Box C_{qs}) \boxtimes C_{pqrs}$ and $(C_{ps} \Box C_{qr}) \boxtimes C_{pqrs}$ are nonisomorphic minimal regular covers of $(C_p \Box C_q) \boxtimes C_{rs}$ and $(C_r \Box C_s) \boxtimes C_{pq}$. The covered graphs are nonisomorphic, isovalent and of the same size.*

Proof. Let $p < q < r < s$ be odd primes, $H_1 = (C_p \Box C_q) \boxtimes C_{rs}$ and $H_2 = (C_r \Box C_s) \boxtimes C_{pq}$. Note that both H_1 and H_2 are Cayley graphs. To be more precise, let $Z_k = \langle a_k | a_k^k = 1 \rangle$. Then H_1 is the Cayley graph $\Gamma(G_1, S_1)$ of the group G_1 with respect to the generating set S_1, where $G_1 = Z_p \times Z_q \times Z_{rs}$ and $S_1 = \{a_p, a_q, a_{rs}, a_{rs}a_p, a_{rs}a_p^{-1}, a_{rs}a_q, a_{rs}a_q^{-1}\}$. Similarly, we have $H_2 = \Gamma(G_2, S_2)$, where $G_2 = Z_r \times Z_s \times Z_{pq}$ and $S_2 = \{a_r, a_s, a_{pq}, a_{pq}a_r, a_{pq}a_r^{-1}, a_{pq}a_s, a_{pq}a_s^{-1}\}$.

By the theory of covering spaces any regular covering $\tilde{H}_1$ of a Cayley graph H_1 also is a Cayley graph, say $\tilde{H}_1 = \Gamma(\tilde{G}_1, \tilde{S}_1)$, and there is a normal subgroup N_1 in $\tilde{G}_1$ such that $H_1 = \tilde{H}_1 / N_1$. In other words, H_1 is the quotient graph of $\tilde{H}_1$ under the action of N_1 on $\tilde{H}_1$ by left multiplication.

This also holds for the universal cover U of H_1 that is, $U = \Gamma(G, S)$, where G is the free group generated by a set S of 7 generators. Clearly U is also the universal of H_2 and H_1, H_2 are quotients of U under the action of appropriate normal subgroups K_1, respectively K_2, of G. Since the quotient groups are abelian, both K_1 and K_2 contain the commutator group C of G.

Consider a minimal common cover $\tilde{H}$ of H_1 and H_2. It is the Cayley graph of a group $\tilde{G} = G/\tilde{N}$, where $\tilde{N}$ is a normal subgroup of G. Since $(G/\tilde{N})/(K_1/\tilde{N}) \simeq G/K_1$ we infer $H_1 \simeq \tilde{H}/(K_1/\tilde{N})$. Similarly $H_2 \simeq \tilde{H}/(K_2/\tilde{N})$.

As such $G/\tilde{N}$ need not be normal. In this case N does not contain C and we consider $U/\,\tilde{N}C$, which is an abelian Cayley graph that is definitely not larger than $U/\tilde{N}$. Moreover, since $(G/\,\tilde{N}C)\,/\,(K_1/\,\tilde{N}C) \simeq G/K_1$, the graph $U/\,\tilde{N}C$ is a covering of H_1, and similarly of H_2.

We can therefore restrict our attention to abelian Cayley graphs and show next that the neighborhood graph of every vertex in H_1, H_2 and any minimal common cover are isomorphic. By the neighborhood graph we mean the graph induced by a vertex and its neighbors.

Clearly the neighborhood graphs of H_1 and H_2 are isomorphic. They can be characterized as two strong products $P_3 \boxtimes P_3$, say X_1 and X_2, that are glued together at the center vertex and two nonadjacent vertices of degree 5 (in $P_3 \boxtimes P_3$). We denote it by B. The two edges along which X_1 and X_2 are identified have the highest degrees in B and are thus invariant under all automorphisms of B. Let Y be the path in B consisting of these two edges.

The diameter of B is two. If any cycle in a copy of B, say in H_1, lifts to a path in a minimal common cover $\tilde{H}$ of H_1 and H_2, then there are two vertices of distance two in $\tilde{H}$ that project into the same vertex by the covering projection. Since $\tilde{H}$ is a Cayley graph of a group $\tilde{G}$ this implies that $\tilde{G}$ has an element of even order. Since neither G_1 nor G_2 have elements of even order, an element of even order ≥ 2 will have to be factored out in both projections, contrary to the minimality of the projection.

By the same argument all squares will be lifted to squares.

Thus the neighborhood graphs in H_1, H_2 and any minimal common cover $\tilde{H}$ are isomorphic to B and the covering projections isomorphically map the neighborhood graphs of $\tilde{H}$ onto those of G_1 and G_2. Thus the paths Y that are uniquely defined for every neighborhood graph are also mapped onto the paths Y in G_1 and G_2.

In G_1 these paths form a collection of cycles of length rs, in G_2 a collection of cycles of length pq, and thus a collection of cycles whose length is a multiple of $pqrs$ in $\tilde{H}$.

We now note that both X_1 and X_2 have another path of two edges, different from P, that connects vertices of degrees 5, 8 and 5. We call them Y_1 and Y_2, respectively. Together the collection of paths Y_1 and Y_2 form a collection of disjoint products $C_p \square C_q$ in G_1 and $C_r \square C_s$ in G_2. They are covered by the connected subgraphs formed by the union of the Y_1 and Y_2 in $\tilde{H}$, which cover both the $C_p \square C_q$ and the $C_r \square C_s$. As we know the minimal common covers of these graphs have $pqrs$ vertices, so the number of vertices of $\tilde{H}$ is $(pqrs)^2$.

We are not entirely through yet, because we have not shown yet that H_1 and H_2 are Cayley graphs of no other groups but G_1, respectively G_2. To see this we invoke the structure of the automorphism group of strong and Cartesian products. We first note that $C_x \square C_y$ is prime with respect to the strong product, just as C_z for any $x, y, z > 2$. Moreover, no two adjacent vertices of in any of these graphs have the same neighbors. Under these conditions the automorphism group of $(C_x \square C_y) \boxtimes C_z$ is the direct product of the groups of the factors, see [3, Corollary 5.24]. Similarly, see [3, Corollary 4.17], the automorphism group of $C_x \square C_y$ is the direct product of the groups of the factors, if $x \neq y$. Since a graph is a Cayley graph if and only if it has a subgroup of the automorphism group that acts transitively and fixed point freely we have to find all such groups now. They must be direct products of subgroups of the factors C_k, and these subgroups must act transitively and fixed-point freely on the C_k. Since the k are prime, the only possibilities are the groups Z_k. Thus the groups G_1 and G_2 with respect to which P_1 and P_2 are Cayley graphs are uniquely defined. This is all we needed, we did not need uniqueness of the S_1 and S_2 in terms of the generators of the Z_k.

Thus, our graphs $(C_{pr} \square C_{qs}) \boxtimes C_{pqrs}$ and $(C_{ps} \square C_{qr}) \boxtimes C_{pqrs}$ are indeed non-isomorphic minimal regular common covers of P_1 and P_2. □

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