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# Endomorphisms of free groups and their fixed points

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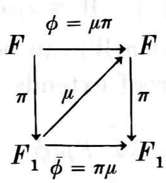
Much work recently has been focused on the issue of fixed words for automorphisms of free groups – see [2] and papers referred to there. Bestvina and Handel [1] have announced a powerful structure theorem for automorphisms of free groups that has as a consequence a proof of what has become known, to the amusement of Peter Scott, as the '*so-called*' *Scott Conjecture*: if $F$ is a free group of rank $n$ and $\alpha: F \to F$ is an automorphism, then $\mathrm{Fix}(\alpha) = \{w \in F \mid \alpha(w) = w\}$ has rank at most $n$.

The purpose of this note is to generalize this to endomorphisms. We also show how to reduce constructively the problem of calculating fixed words – finite or infinite – for an endomorphism to that for an associated monomorphism. Throughout we denote endomorphisms, monomorphisms, automorphisms and free factor projections respectively with the letters $\phi$, $\mu$, $\alpha$ and $\pi$.

THEOREM 1. *If $F$ is a free group of rank $n$ and $\phi: F \to F$ an endomorphism, then there is a $\phi$-invariant subgroup $\phi^{\infty}(F)$ of rank at most $n$ on which $\phi$ is an automorphism and which contains* $\mathrm{Fix}(\phi)$.

COROLLARY. *The 'so-called' Scott Conjecture holds for endomorphisms.*

THEOREM 2. *With $\phi$ and $F$ as above, there exist a constructible free factor decomposition $F = F_1 * F_2$ with projection $\pi: F \to F_1$ and a monomorphism $\mu: F_1 \to F$ so that $\phi = \mu\pi$ giving a commutative diagram*




*and satisfying*

$$\mathrm{Fix}(\phi) \cong \mathrm{Fix}(\bar{\phi}); \quad Fp(\phi) \cong Fp(\bar{\phi}).$$

*Note.* $\bar{\phi}$ is defined by the diagram. $Fp(\phi)$ denotes the set of all fixed words, both finite and infinite: see the proof of Theorem 2.

Theorem 2 can be applied inductively until $\bar{\phi}$ is a monomorphism.

*Proof of Theorem* 1. Let

$$\phi^{\infty}(F) = \bigcap_{m=1}^{\infty} \phi^{m}(F).$$