

Fixed subsets of homomorphisms of free groups*

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Abstract

We derive a bound for the rank of the fixed point set $\text{Fp}(\alpha)$ of monomorphisms α and the fixed point group $\text{Fix}(\alpha)$ of homomorphisms α of subgroups G of a free group F into F . We do not require the ranks of G and F to be finite; these conditions are replaced by a finiteness condition for α .

1 Introduction

S. M. Gersten [4] showed that the fixed point subgroup $\text{Fix}(\alpha)$ of an automorphism α of a finitely generated free group F is itself finitely generated. D. Cooper [3] proved that the set $\text{Fp}(\alpha)$ of fixed points and fixed ends of α is also finitely generated. We derive bounds for the ranks of $\text{Fix}(\alpha)$ and $\text{Fp}(\alpha)$, which also hold for mappings which are not necessarily automorphisms of finitely generated free groups, by extending a technique of R. Z. Goldstein and E. C. Turner [5].

These bounds depend on α and may be arbitrarily large, also in case of automorphisms of free groups F of finite rank. For this case, however, it is conjectured (see J. R. Stallings [9]) that $\text{rank } \text{Fix}(\alpha) \leq \text{rank } F$.

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M. Bestvina [1] and Handel have announced a powerful structure theorem for automorphisms of free groups which has as consequence a proof of this conjecture. In spite of this achievement it still seems to be unknown whether the maximal length of generators in a Nielsen reduced generating set of $\text{Fix}(\alpha)$ is a recursive function of α .

However, if one is satisfied with finding all generators of a Nielsen reduced generating set of $\text{Fix}(\alpha)$ which have length $\leq N$ one can determine them with an algorithm essentially quadratic in N (see [7]). A variant of this algorithm determines all generators if α is a so-called positive automorphism of a free group F , i.e. if all letters in the reduced form of every $\alpha(s)$, $s \in S$, where S freely generates F , have positive exponents. (See also [2].)

2 Graphs

To fix notation we recall that a (*directed*) *graph* X is a collection of two sets V and E , together with two mappings $o, t : E \rightarrow V$. The set V is the *vertex set* and E the *edge set*. Let $y \in E$. Then $o(y)$ is called the *origin* and $t(y)$ the *terminus* of the *edge* y .

The *out-degree* $d^+(a)$ of a vertex a is the number of edges originating in a and the *in-degree* $d^-(a)$ is the number of edges terminating in a , whereas $d(a) = d^+(a) + d^-(a)$ is called the *degree* of a .

For formal reasons it is useful to introduce a set E^{-1} , disjoint from E , together with a bijection $y \mapsto y^{-1}$ of E onto E^{-1} . For the inverse of this bijection we shall use the same notation. Hence $y \mapsto y^{-1}$ becomes an involution on $E \cup E^{-1}$. We also extend o and t to E^{-1} by setting

$$o(y^{-1}) = t(y) \quad \text{and} \quad t(y^{-1}) = o(y).$$

The symbol y^{-1} is introduced to indicate that the edge y is traversed from $t(y)$ to $o(y)$, as will become clear by the definition of a walk.

By a *walk* w from a vertex $a = o(w)$ to a vertex $b = t(w)$ we mean a sequence

$$w = y_1 y_2 \dots y_k$$

of elements y_i in $E \cup E^{-1}$ subject to the conditions $a = o(y_1)$, $t(y_i) = o(y_{i+1})$ for $1 \leq i < k$ and $t(y_k) = b$. We say w *connects* a *with* b . By admitting $k = 0$ connectedness becomes an equivalence relation on V . It is clear what we mean by a *connected* graph.

The walk w is *reduced* if $y_i \neq y_{i+1}^{-1}$ for $1 \leq i < k$. We note that every walk w contains a uniquely defined reduced subwalk from a to b . w is *closed* if $o(w) = t(w)$ and a reduced closed walk is a *circuit*. If w is reduced, closed and $y_1 \neq y_k^{-1}$ we speak of a *cyclically reduced closed walk* or a *cycle*. An acyclic, connected graph is called a *tree*.

A walk w is called *uniformly directed*, if $y_i \in E$ for $1 \leq i \leq k$.

We shall also admit *one-sided infinite walks*

$$y_1 y_2 y_3 \dots$$

and *two-sided infinite walks*

$$\dots y_{-2} y_{-1} y_0 y_1 y_2 \dots$$

Reduced one-sided infinite walks will also be called *rays* and two-sided infinite ones *two-sided rays*

Two one-sided infinite walks u, v are called *equivalent*, if there is a third one, which meets both u and v infinitely often. This relation is an equivalence relation. The equivalence classes with respect to this relation are called *ends*.

A *sink* of a graph is a vertex a with $d^+(a) = 0$ and an end e , such that every ray $u \in e$ has only finitely many edges in E^{-1} is called a *sink at infinity*. In this case an end-segment of u is uniformly directed towards e .

We say a walk u ends in a subgraph Y or an end e of X if an end-segment of u is contained in Y or e .

The following easy lemma describes a situation we often have to deal with. It can be considered as a relationship between forests and mappings. (See V. F. Kolchin [8, Chapter 3.1] for the case of finite graphs.)

Lemma 1 *Let X be a connected graph, each vertex of which has outdegree ≤ 1 . Then X has exactly one cycle or is a tree with exactly one sink. Every maximal uniformly directed walk of X ends in the sink or the cycle of X .*

Proof. Since $d^+(x) \leq 1$ for all $x \in V(X)$ there exists to every $y \in V(X)$ a unique, uniformly directed walk w_y originating in y .

If w_y has finite length, possibly zero, it ends in a sink.

If w_y meets itself, let n be the smallest index to which there exists an index $m < n$ with $y_m = y_n$. Then

$$w_y = y_1 \dots y_{m-1} y_m \dots y_{n-1} y_m \dots y_{n-1} \dots$$

and w_y ends in the cycle $y_m \dots y_{n-1}$.

For the remainder of the proof, let a be a fixed but arbitrary vertex, b any other vertex of X , and

$$v = u_1 \dots u_k$$

be a shortest walk from b to w_a , i.e. $o(u_1) = b$, $u_k \notin w_a$ but

$$t(u_k) \in \{a, t(y_i) \mid i \leq 1\}.$$

Since $t(y_i)$ is either a sink or the origin of $y_{i+1} \neq u_k$, $y_{i+1} \in E$, it is clear that $u_k \in E$. But then u_{k-1}, u_{k-2}, \dots are also in E . Thus, v is uniformly directed and therefore an initial segment of w_b . As b was chosen arbitrarily, this means that every $w_x, x \in V(X)$, ends in w_a .

If w_a ends in a cycle, all w_x must end in this cycle and X cannot contain any other cycles.

If w_a is finite, every w_x must end in the endpoint of w_a and this endpoint is the only sink of X , finite or infinite.

Finally, if w_a is infinite and does not end in a cycle X must be a tree and any w_x differs from w_a in at most finitely many edges. Thus, the end defined by w_a is a sink at infinity. \square

3 Cayley Graphs

The *Cayley graph* $\Gamma(G, S)$ of a group G with respect to a subset S of G is defined on the vertex set G with the edge set $G \times S$. The initial vertex $o(g, s)$ of an edge (g, s) is g and the terminal one gs . The set E^{-1} is adjoined as usual. Note that $\Gamma(G, S)$ is connected iff S generates G .

However, if $S \cap S^{-1} = \emptyset$ the element (gs, s^{-1}) is not in E and we can identify (gs, s^{-1}) with $(g, s)^{-1}$, in particular this is possible when G is a free group freely generated by S .

For any subgroup H of G the *coset graph* $\Gamma(G, S)/H$ is defined on the vertex set

$$\{Hg \mid g \in G\}$$

and the edge set

$$\{(Hg, s) \mid s \in S\},$$

where o, t and $^{-1}$ are introduced such that the mapping

$$g \mapsto Hg, (g, s) \mapsto (Hg, s)$$

becomes a homomorphism of graphs.

If X is a connected Cayley graph $\Gamma(G, S)$ the mapping

$$\psi : (g, s) \mapsto s$$

of $E(X)$ into G naturally extends to a homomorphism of the fundamental group πX onto G .

For $Y = \Gamma(G, S)/H$ the mapping

$$\psi : (Hg, s) \mapsto s$$

analogously extends to a mapping of πY into G . If G is free ψ is an isomorphism of πY onto H .

Furthermore, the relation

$$t(w) = o(w)\psi(w)$$

holds for any walk w in X or Y .

In case G is a free group freely generated by S the quotient graph $\Gamma(G, S)/H$ is the covering space corresponding to the bouquet of $|S|$ loops (i.e. $\Gamma(G, S)/G$) corresponding to the subgroup H of $\pi(\Gamma(G, S)/G)$.

Lemma 2 (Goldstein and Turner [5]) *Let G be a subgroup of a free group F , α be a homomorphism of G into F and $\text{Fix}(\alpha)$ be the fixed-point subgroup of α , i.e.*

$$\text{Fix}(\alpha) = \{g \mid \alpha(g) = g, g \in G\}.$$

Choose a free generating set S of G and let Z be the graph defined on the vertex-set F with the edge-set $F \times S$, where

$$o_Z(f, s) = f \quad \text{and} \quad t_Z(f, s) = \alpha(s)^{-1}fs.$$

Then the graph $Y = \Gamma(G, S)/\text{Fix}(\alpha)$ is isomorphic to the component of Z containing 1.

Proof. Set $H = \text{Fix}(\alpha)$ and define φ by

$$\varphi(Hg) = \alpha(g)^{-1}g \quad \text{and} \quad \varphi(Hg, s) = (\alpha(g)^{-1}g, s).$$

Clearly φ is well-defined, because for $Hg = Hg_1$ we have $g = hg_1$ and thus

$$\alpha(g)^{-1}g = \alpha(hg_1)^{-1}hg_1 = \alpha(g_1)^{-1}\alpha(h)^{-1}hg_1 = \alpha(g_1)^{-1}g_1.$$

Conversely, $\alpha(g)^{-1}g = \alpha(g_1)^{-1}g_1$ implies $g_1g^{-1} \in H$, i.e. $Hg = Hg_1$. Thus φ is injective.

That φ is a homomorphism follows from

$$\varphi(o_Y(Hg, s)) = \varphi(Hg) = \alpha(g)^{-1}g = o_Z(\alpha(g)^{-1}g, s)$$

and

$$\varphi(t_Y(Hg, s)) = \varphi(Hgs) = \alpha(gs)^{-1}gs = t_Z(\alpha(g)^{-1}g, s).$$

Finally, define

$$\psi(f, s) = s \quad \text{and} \quad \psi(f, s)^{-1} = s^{-1}$$

for $s \in S$. Then ψ extends to the fundamental groupoid of Z and for any walk w in Z we have

$$t(w) = \alpha(\psi w)^{-1}o(w)\psi w.$$

Given any $g = s_1s_2 \dots s_n$ in G there is a w in Z originating in 1 with $\psi w = g$. Hence the component of Z containing 1 consists of all vertices of the form $\alpha(g)^{-1}g$, i.e. of the images of the vertices of Y . \square

Corollary 1 (See also Cohen and Lustig [2]) *Let $f \in F$ and define $\beta : G \rightarrow F$ by*

$$\beta(g) = f^{-1}\alpha(g)f.$$

Then the component of Z containing f is isomorphic to $\Gamma(G, S)/\text{Fix}(\beta)$.

Proof. Clear.

4 Rank of the fixed point subgroup

Let F be a free group freely generated by A and let G be a subgroup of F . A Nielsen-reduced basis of G is a free generating set $S = \{s_\iota \mid \iota \in I\}$ of G with the following properties:

(a) Every s_ι is of the reduced form $s_\iota = o_\iota a_\iota t_\iota$, where o_ι and t_ι are words in $A \cup A^{-1}$ and $a_\iota \in A \cup A^{-1}$.

(b) The elements a_ι, a_κ are never cancelled in a product $s_\iota s_\kappa^{\pm 1}$ unless it is of the form $s_\iota s_\iota^{-1}$.

Theorem 1 *Let G be a subgroup of a free group F freely generated by A and let*

$$S = \{o_\iota a_\iota t_\iota \mid \iota \in I\}$$

be a Nielsen-reduced generating set of G . Suppose α is a homomorphism of G in F and let n_ι be the number of times an endsegment of $o_\iota a_\iota$ is a subword of the freely reduced form of $\alpha(o_\iota a_\iota t_\iota)$.

Then $\sum_{\iota \in I} n_\iota$ bounds the rank of the fixed point group of α .

Proof. Let $X = \Gamma(G, S)$, $Y = X/\text{Fix}(\alpha)$ and Z be defined as before. Denoting the component of Z containing 1 by Z_1 we recall that $Z_1 \cong Y$ and that $\pi(Y) \cong \text{Fix}(\alpha)$. To find a bound for the rank of $\text{Fix}(\alpha)$ it therefore suffices to find a bound for the rank of πZ_1 .

Setting $s_\iota = o_\iota a_\iota t_\iota$ for $\iota \in I$ we note that the edges of Z are of the form

$$(f, s_\iota)$$

with the endpoints f and $\alpha(s_\iota)^{-1} f s_\iota$. If $o_\iota a_\iota$ is cancelled in the product $f s_\iota$ we say there is *large cancellation* in $f s_\iota$. Analogously we say there is large cancellation in $f s_\iota^{-1}$ if $t_\iota^{-1} a_\iota^{-1}$ is cancelled in $f s_\iota^{-1}$. We now reorient the edges of Z as follows:

If there is large cancellation in $f s_\iota$ we define

$$o_n(f, s_\iota) = f$$

and if there is large cancellation in $(\alpha(s_\iota)^{-1} f s_\iota) s_\iota^{-1}$ we set

$$o_n(f, s_\iota) = \alpha(s_\iota)^{-1} f s_\iota.$$

Obviously large cancellation in $f s_\iota$ excludes large cancellation in $f s$, $s \in S \cup S^{-1} \setminus \{s_\iota\}$. Thus $d_n^+(f) \leq 1$ and $d_n^+(1) = 0$.

It should be noted, that some edges may be oriented in both directions this way. If this is the case, we arbitrarily choose one orientation. Furthermore, the edges (f, s_ι) with no large cancellation in $f s_\iota$ and $(\alpha(s_\iota)^{-1} f s_\iota) s_\iota^{-1}$ will remain unoriented. This is only possible if the reduced form of $\alpha(s_\iota)^{-1} f s_\iota$ does not end in $a_\iota t_\iota$ although $f s_\iota$ ends in $a_\iota t_\iota$.

Suppose $o_l^* a_l$ is the largest endsegment of $o_l a_l$ such that $f s_l$ ends in $a_l t_l$, i.e. $f s_l$ is of the form $f^* o_l^* a_l t_l$. Then (the reduced form of) $\alpha(s_l)$ must begin with $f^* o_l^* a_l$. Hence (f, s_l) can remain unoriented only if an endsegment of $o_l a_l$ is a subword of $\alpha(s_l)$.

To see that every such subword can give rise to at most one unoriented edge suppose $a_l b$ is an endsegment of $\alpha(s_l)$ such that

$$\alpha(s_l) = f^* o_l^* a_l b,$$

reduced as written. Any f with large cancellation in $f s_l$ and large cancellation in $(\alpha(s_l)^{-1} f s_l) s_l^{-1}$ must be of the reduced form $f^{**} (o_l^{**})^{-1}$, where o_l^{**} is an initial segment of o_l , such that

$$b^{-1} a_l^{-1} (o_l^*)^{-1} (f^*)^{-1} f^{**} (o_l^{**})^{-1} o_l a_l t_l$$

does not end in $a_l t_l$. Suppose the a_l preceding t_l cancels against the a_l^{-1} following b^{-1} . Then

$$(o_l^*)^{-1} (f^*)^{-1} f^{**} (o_l^{**})^{-1} o_l$$

must freely reduce to 1. Setting $o_l^{***} = o_l (o_l^{**})^{-1}$ we thus have

$$f^* o_l^* = f^{**} o_l^{***},$$

where both sides are reduced as written. Thus o_l^{***} is an endsegment of o_l^* with

$$\alpha(s_l) = f^{**} o_l^{***} a_l b$$

and

$$(f^{**} (a_l^{**})^{-1}, s_l)$$

is unoriented. Let Z^* be this new graph.

Thus $N = \sum_{l \in I} n_l$ is an upper bound on the number of unoriented edges. It remains to show that N bounds the rank of $\pi(Z_1^*)$. For any maximal subtree T of Z_1^* this number is the number of edges of Z_1^* not in T .

To see this, let U_1 be the set of unoriented edges in Z_1^* . Then the graph W obtained from Z_1^* after removal of all edges in U_1 consists of components every vertex of which has outdegree ≤ 1 . By Lemma 1 every such component has cyclomatic number ≤ 1 . We also note that the component of 1 in W , which contains a sink, has cyclomatic number 0.

Suppose U_1 is infinite. Clearly W can have at most $|U_1|$ components. Removal of at most one edge in every component of W leaves an acyclic subgraph of W which contains all but at most $2|U_1| = |U_1|$ edges. Thus N bounds the rank of $\text{Fix}(\alpha)$ in this case.

If U_1 is finite it is clear that Z_1^* contains a finite connected subgraph W_1 containing 1 and all cycles of Z_1^* . Let n be the number of vertices of W_1 . Since no oriented edge originates in 1 and at most one at every other vertex, W_1 contains at most $n - 1$ oriented edges i.e. at most $n - 1 + |U_1|$ edges altogether. Since every spanning tree of W_1 has $n - 1$ vertices, the rank of W_1 is bounded by $|U_1|$, and thus by N . \square

5 Endomorphisms and automorphisms of free groups

A special case of Theorem 1 is the following theorem, which is also contained in [2] for automorphisms and in the cited form in [6]:

Theorem 2 *Let G be a free group freely generated by $S = \{s_\iota \mid \iota \in I\}$, α be an endomorphism of G and let n_ι be the number of occurrences of s_ι^{+1} in $\alpha(s_\iota)$, written as a reduced word in $S \cup S^{-1}$. Then the rank of $\text{Fix}(\alpha)$ is bounded by $\sum_{\iota \in I} n_\iota$.*

In this case it is quite easy to characterize the case when edges of Z are unoriented resp. oriented both ways:

Lemma 3 *Let G, S, α be given as in Theorem 2 and let wsv^{-1} be the reduced form of $\alpha(s)$, $s \in S$. Then the pair of (not necessarily distinct) vertices $\{w, v\}$ is connected by an unoriented edge.*

If $\alpha(s) = ws^{-1}v^{-1}$ the pair $\{ws^{-1}, vs\}$ is connected by an edge oriented both ways.

Proof. Clear.

Lemma 4 *Under the assumptions of Theorem 2, let $\alpha(s) = gs^n as^{-n} g^{-1}$, $s \in S$. Then s^n contributes at most one to the rank of $\text{Fix}(\alpha)$.*

Proof. We note first that any edge in Z_1^* which is oriented both ways allows us to reduce the rank by 1. Hence, if gs^i , $1 \leq i < n$, is in Z_1^* , then so is $gs^n a s^{n-i}$, which is connected with gs^i by an unoriented edge. This implies that the pair

$$\{gs^n a s^{n-i+1} s^{-1}, gs^{-i+1}\},$$

connected by a doubly oriented edge, is also in Z_1^* . \square

6 Bounded cancellation

Let α be a monomorphism of a subgroup G of a free group F into F , where F is freely generated by A and G by S . If $f \in F$, let $|f|_F$ denote the length of f written as a reduced word in $A \cup A^{-1}$ and for $g \in G$ let $|g|_G$ denote the length of g as a reduced word in $S \cup S^{-1}$. (If there is no danger of confusion we shall omit the subscript in either case.)

The common initial segment of two words w_1 and w_2 in $S \cup S^{-1}$ will be denoted by $w_1 \wedge_F w_2$. Analogously we define \wedge_G . Again we shall frequently omit subscripts. If w_1 already is an initial segment of w_2 we write $w_1 \nearrow_F w_2$ or $w_1 \nearrow w_2$.

We say α has the *bounded cancellation property* if there exists a constant N_α such that the length $|\alpha(w_1) \wedge_F \alpha(w_2)|$ of the common initial segment of $\alpha(w_1), \alpha(w_2)$ for elements $w_1, w_2 \in G$ is less than N_α if $w_1 \wedge_G w_2 = 1$.

D. Cooper [3] showed that automorphisms of finitely generated free groups F have the bounded cancellation property.

Theorem 3 *Let F be a free group freely generated by A , G be a subgroup freely generated by S and let α be a monomorphism from G into F . If*

$$L = 2 \sum_{s \in S} (|\alpha(s)| - 1)$$

is finite, then α has the bounded cancellation property with a cancellation bound $N_\alpha \leq lL^2$, where $l = \max_{s \in S} |\alpha(s)|$.

Proof. Let $w_1 = s_1 \dots s_m$ and $w_2 = t_1 \dots t_n$ be two elements of G , written as reduced words in $S \cup S^{-1}$ with $w_1 \wedge w_2 = 1$, i. e. $s_1 \neq t_1$.

Set $a = \alpha(w_1) \wedge \alpha(w_2)$. We wish to show that $|a| \leq lL^2$.

Suppose $1 \neq b \nearrow \alpha(s_1 s_2 \dots s_m) \wedge \alpha(t_1 t_2 \dots t_n) = a$ and let i, j be the smallest indices such that $b \nearrow \alpha(s_1 s_2 \dots s_i) \wedge \alpha(t_1 t_2 \dots t_j)$.

We observe that $i, j \neq 1$ since b is not the empty word and that b is on a shortest path from $\alpha(s_1 s_2 \dots s_{i-1})$ to $\alpha(s_1 s_2 \dots s_i)$ as well as from $\alpha(t_1 t_2 \dots t_{j-1})$ to $\alpha(t_1 t_2 \dots t_j)$. Hence $\alpha(s_1 s_2 \dots s_i)^{-1} \alpha(t_1 t_2 \dots t_j)$ consists of an initial segment of $\alpha(s_i)^{-1}$ and a terminal segment of $\alpha(t_j)$. The number

of possibilities for each of these segments being L it is clear that we have at most L^2 possibilities for $\alpha(s_1 s_2 \cdots s_i)^{-1} \alpha(t_1 t_2 \cdots t_j)$, and therefore also for $w_b = s_i^{-1} \cdots s_1^{-1} t_1 \cdots t_j$.

If $c \nearrow b \nearrow a$ and the distance between c and b is at least l it is easy to see that w_c is a proper subword of the reduced word w_b and therefore different from w_b . But then $|a|$ must be smaller than lL^2 , for otherwise we could find a sequence of elements

$$1 \neq b_1 \nearrow b_2 \nearrow \cdots \nearrow b_{L^2} \nearrow b_{L^2+1} = a,$$

any two of which would have distance at least l from each other, whence no two w_{b_i} could be the same, in contradiction to the fact that there are at most L^2 different possibilities for these elements. \square

Corollary 2 *The mapping $\alpha^{-1}: \alpha G \rightarrow G$ has the bounded cancellation property if $\sum_{s \in S} (|s| - 1)$ is finite.*

Corollary 3 *If w is a reduced word in $S \cup S^{-1}$ of length $\geq nL^2$, then $|\alpha(w)| > n$.*

Corollary 4 *Let α be a monomorphism of the finitely generated free group $F = \langle S \mid \ \rangle$ of rank r into itself and let l be defined as above. Then α has a cancellation bound $N_\alpha \leq 4r^2 l^3$.*

Lemma 5 *Let $\alpha : G \rightarrow F$ be given as above and let N_α be the cancellation bound of α . If w_1 is an initial segment of the reduced word w_2 in $S \cup S^{-1}$, then*

$$|\alpha(w_1) \wedge \alpha(w_2)| \geq |\alpha(w_1)| - N_\alpha.$$

Proof. Since $w_1^{-1} \wedge w_1^{-1} w_2 = 1$ we have

$$|\alpha(w_1^{-1}) \wedge \alpha(w_1^{-1} w_2)|_F \leq N_\alpha.$$

Set $u = \alpha(w_1) \wedge \alpha(w_2)$ and $v = \alpha(w_1^{-1}) \wedge \alpha(w_1^{-1} w_2)$. Clearly

$$\alpha(w_1) = uv^{-1}$$

$$|\alpha(w_1)| = |u| + |v|.$$

This proves the lemma. \square

7 Action on infinite words

Let α be a homomorphism of a subgroup G of a free group F into F , where F is freely generated by A and G by S . Let $s_1s_2\dots$ be an infinite, reduced word in $S \cup S^{-1}$ and $a_1a_2\dots$ be one in $A \cup A^{-1}$. We say

$$\alpha(s_1s_2\dots) = a_1a_2\dots$$

if for every index k there exists a K such that

$$a_1a_2\dots a_k \not\prec \alpha(s_1s_2\dots s_n) \text{ for } n > K.$$

If α is the identity mapping and

$$\alpha(s_1s_2\dots) = a_1a_2\dots,$$

then we say that $s_1s_2\dots$ can be represented as the infinite, reduced word $a_1a_2\dots$ in $A \cup A^{-1}$.

D. Cooper [3] showed that the action of automorphisms of finitely generated free groups extends to infinite, reduced words.

Lemma 6 *Let F be a free group freely generated by A , G be a subgroup freely generated by S and let α be a monomorphism from G into F . If*

$$\sum_{s \in S} (|\alpha(s)| - 1)$$

is finite, then α can be extended to infinite, reduced words in $S \cup S^{-1}$.

Proof. We show first that the words

$$w_i = \alpha(s_1s_2\dots s_i)$$

have unbounded lengths. If the set $\{s_1, s_2, \dots\}$ of letters used in $s_1s_2\dots$ is finite this is obvious. If this set is infinite, then infinitely many different s with $|\alpha(s)| = 1$ must occur. If there is a constant C such that $|w_i| \leq C$ for all i , then almost all of these letters have to be cancelled again. But this is only possible with infinitely many different $t \in S \cup S^{-1}$ with $|\alpha(t)| \geq 2$.

Thus, there is a function $M(m)$ such that

$$|w_i| > m \text{ for } i > M(m).$$

Let N_α be the cancellation bound for α . Set

$$K(k) = M(k + N_\alpha)$$

and consider $w_i \wedge w_j$ as a word in $A \cup A^{-1}$ for $i, j > K(k)$. By Lemma 5

$$|w_i \wedge w_j| \geq k + N_\alpha - N_\alpha.$$

Let $a_1 \dots a_k$ be defined as the initial segment of length k in $w_i \wedge w_j$. Then $a_1 \dots a_k \nearrow s_1 \dots s_i$ for any $i > K(k)$ and

$$\alpha(s_1 s_2 \dots) = a_1 a_2 \dots,$$

which had to be proved. \square

Corollary 5 *Let G be a subgroup of a free group F , where F is freely generated by A and G by S . If*

$$\sum_{s \in S} (|s| - 1)$$

is finite, then every infinite, reduced word in $S \cup S^{-1}$ can be represented as an infinite, reduced word in $A \cup A^{-1}$.

Proof. Let α be the identity on G and apply Lemma 6. \square

Note that the infinite words of F correspond to the ends of $\Gamma(F, A)$. For, let e be an end of $\Gamma(F, A)$ and $w = y_1 y_2 y_3 \dots$ the unique, reduced infinite path in with $o(w) = 1$. Then we associate the infinite word

$$\psi y_1 \psi y_2 \dots$$

with w . Clearly this is a bijection.

If w_1, w_2, \dots is a sequence of elements in F and $w = a_1 a_2 \dots$ an infinite, reduced word in $A \cup A^{-1}$, we say

$$\lim_{i \rightarrow \infty} w_i = w,$$

if

$$\lim_{i \rightarrow \infty} |w \wedge w_i| = \infty.$$

For $\alpha(s_1 s_2 \dots) = a_1 a_2 \dots$ we clearly have

$$\lim_{i \rightarrow \infty} \alpha(s_1 \dots s_i) = a_1 a_2 \dots$$

8 Rank of the fixed point set

Let G be a subgroup of a free group F , where F is freely generated by A and G by S . Furthermore, let α be an automorphisms of F which extends to infinite words on G . Denote the set of infinite, reduced words in $S \cup S^{-1}$ by G^* . Then the fixed point set $\text{Fp}(\alpha|G)$ is defined by

$$\text{Fp}(\alpha|G) = \{w|\alpha(w) = w, w \in G \cup G^*\}.$$

It has been shown by D. Cooper [3] that $\text{Fp}(\alpha)$ is finitely generated if α is an automorphism of a finitely generated free group. We slightly extend this result, give a bound on the rank and relate the basis of $\text{Fp}(\alpha|G)$ with the sinks and cycles in Z_1^* .

Theorem 4 *Let α be an automorphisms of a free group F freely generated by A and let G be an subgroup with a Nielsen-reduced set*

$$S = \{s_\iota | \iota \in I\}$$

of generators $s_\iota = o_\iota a_\iota t_\iota, a_\iota \in A \cup A^{-1}, o_\iota, t_\iota \in F$, and suppose that

$$\sum_{s \in S} (|\alpha^{-1}(s)| + |s| + |\alpha(s)| - 3)$$

is finite. Furthermore, let m_ι be the number of times an endsegment of $o_\iota a_\iota$ is a subword of the freely reduced forms of $\alpha(o_\iota a_\iota t_\iota)$ or $\alpha^{-1}(o_\iota a_\iota t_\iota)$. Then $\sum_{\iota \in I} m_\iota$ bounds the rank of $\text{Fp}(\alpha|G)$.

Proof. We note first that our assumptions imply bounded cancellation for α on G and for α^{-1} on $G \cup \alpha G$. Let N be the maximum of the cancellation bounds for α and α^{-1} .

Furthermore, it is clear that, α and α^{-1} extend to infinite words in $S \cup S^{-1}$.

Let $w = s_1 s_2 \dots$ be a reduced, infinite word in $S \cup S^{-1}$ fixed by α , i.e.

$$\alpha(s_1 s_2 \dots) = s_1 s_2 \dots$$

Setting $w_n = s_1 s_2 \dots s_n$ we note that this is equivalent to

$$\lim_{n \rightarrow \infty} |w_n \wedge \alpha(w_n)| = \infty.$$

Suppose there is a length L which is infinitely often attained by $|\alpha(w_n)^{-1}w_n|$. Then there exists a sequence $\{n_i | i \geq 1\}$ such that all $\alpha(w_{n_i})^{-1}w_{n_i}$ are equal. Set $u = w_{n_1}$. Then $\alpha(w_{n_i})^{-1}w_{n_i} = \alpha(u)^{-1}u$ and $w_{n_i}u^{-1} \in \text{Fix}(\alpha|G)$. Since

$$\lim_{i \rightarrow \infty} w_{n_i}u^{-1} = w$$

w is the limit of elements in $\text{Fix}(\alpha|G)$. In this sense we can say that w is generated by the elements of a generating set of $\text{Fix}(\alpha|G)$.

Now, suppose that

$$\lim_{n \rightarrow \infty} |\alpha(w_n)^{-1}w_n| = \infty$$

and let K be an index, such that

$$|\alpha(w_n)^{-1}w_n| > N + 3 \max_{s \in S} (|s|, |\alpha(s)|) \quad (1)$$

for $n > K$.

Let $c_n = w_n \wedge \alpha(w_n)$ and consider $c_n^{-1}w_n$ and $c_n^{-1}\alpha(w_n)$. Since

$$|\alpha(w_n)^{-1}w_n| = |c_n^{-1}w_n| + |c_n^{-1}\alpha(w_n)|$$

at least one of the sets $\{|c_n^{-1}w_n| | n \geq 1\}$ and $\{|c_n^{-1}\alpha(w_n)| | n \geq 1\}$ has to be unbounded.

Set $s = o_s a_s t_s$ for $s \in S \cup S^{-1}$ and suppose that for some $n > K$

$$|c_n^{-1}w_n| > |t_{s_n}| \quad (2)$$

and

$$|c_n^{-1}\alpha(w_n)| > N \quad (3)$$

hold. The first condition implies that

$$c_n = w_n s_{n+1} \wedge \alpha(w_n)$$

and a fortiori

$$c_n = w_{n+k} \wedge \alpha(w_n)$$

for $k \geq 0$.

This contradicts the existence of a k with

$$|c_n| < |c_{n+k}|,$$

since $|c_i| \rightarrow \infty$. Hence (2) and (3) cannot both hold for $n > K$.

Case 1. Suppose (3) holds for some $n > K$. Then

$$|c_n^{-1}w_n| \leq |t_{s_n}|.$$

Hence,

$$|c_{n+1}| \leq |w_{n+1}| \leq |c_n| + |t_{s_n}| + |s_{n+1}|$$

and

$$|\alpha(w_{n+1})| \geq |\alpha(w_n)| - |\alpha(s_{n+1})|.$$

By (1) this implies

$$\begin{aligned} |\alpha(w_{n+1})| &> |c_n| + N + 2 \max_{s \in S} (|s|, |\alpha(s)|) \\ &\geq |c_{n+1}| + N. \end{aligned}$$

Therefore

$$|c_{n+1}^{-1}\alpha(w_{n+1})| > N$$

and

$$|c_{n+1}^{-1}w_{n+1}| \leq |t_{s_{n+1}}|.$$

This means large cancellation in

$$(\alpha(w_n)^{-1}w_n)s_{n+1}.$$

By induction we see that large cancellation holds for all

$$(\alpha(w_{n+k})^{-1}w_{n+k})s_{n+k+1}.$$

This implies that the edge

$$(\alpha(w_{n+k})^{-1}w_{n+k}, s_{n+k+1})$$

is oriented in Z_1^* from $\alpha(w_{n+k})^{-1}w_{n+k}$ towards $\alpha(w_{n+k+1})^{-1}w_{n+k+1}$ for any $k > 0$. Thus, the walk in Z_1^* following the vertices

$$\alpha(w_n)^{-1}w_n, \alpha(w_{n+1})^{-1}w_{n+1}, \dots$$

is a uniformly directed, reduced infinite path.

Such walks correspond to the sinks at infinity of Z_1^* , of which there are at most $\sum_{i \in I} n_i$, where n_i is the number of times an endsegment of $o_i a_i$ is a subword of $\alpha(o_i a_i t_i)$.

For every sink e of Z_1^* choose a reduced one-sided infinite path p originating in 1 and ending in e . Let P be the set of these paths and suppose q is the path cofinal with

$$\alpha(w_n)^{-1}w_n, \alpha(w_{n+1})^{-1}w_{n+1}, \dots$$

Then there must be an element $\alpha(u)^{-1}u$ in q from which on these paths are identical, i. e.

$$\alpha(u)^{-1}u = \alpha(w_m)^{-1}w_m$$

for some m . We now note that

$$\alpha(w_m u^{-1})^{-1}w_m u^{-1} = 1,$$

i. e. $w_m u^{-1} \in \text{Fix}(\alpha|G)$. Also,

$$\psi q = u s_{m+1} s_{m+2} \dots$$

Hence, $\text{Fix}(\alpha|G)$ and ψq generate $w = s_1 s_2 \dots$.

Case 2. We have just seen that (3) holds for all $n > K$ or for none. Thus, we still have to consider the case

$$|c_n^{-1} \alpha(w_n)| \leq N$$

for all $n > K$.

Let

$$d_n = |w_n \wedge \alpha^{-1}(w_n)|.$$

It is not hard to see that, because of bounded cancellation and Corollary 3,

$$|d_n^{-1} w_n| \leq \text{const.}$$

Replacing the rôle of α and α^{-1} this clearly implies that the distances $|d_n^{-1} \alpha^{-1}(w_n)|$ must be unbounded. This corresponds to Case 1. \square

For automorphisms of finitely generated free groups we obtain:

Theorem 5 (See also [6]) *Let F be a free group freely generated by s_1, \dots, s_n and let α be an automorphism of F . Set m_i for the number of occurrences of s_i in $\alpha(s_i)$ and $\alpha^{-1}(s_i)$. Then the rank of $\text{Fp}(\alpha)$ is bounded by $\sum_{i=1}^n m_i$.*

Proof. Clear. \square

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