FACTORING CARTESIAN-PRODUCT GRAPHS

Wilfried Imrich
Institut für Mathematik und Angewandte Geometrie
Montanuniversität, A-8700 Leoben
Austria

Janez Žerovnik [1]
Inštitut za matematiko, fiziko in mehaniko
Jadranska 19, 61111 Ljubljana
Slovenia

Abstract

In a fundamental paper Sabidussi [Math.Z. 72 (1960), 446-457] used a tower of equivalence relations on the edge set $E(G)$ of a connected graph G to decompose G into a Cartesian product of prime graphs. Later a method by Graham and Winkler [Trans.Amer.Math. Soc. 288 (1985), 527-533] of embedding a connected graph isometrically into Cartesian products opened another approach to this problem. In both approaches an equivalence relation σ which determines the prime factorization is constructed. The methods differ by the starting relations used. We show that σ can be obtained as the convex hull of the starting relation used by Sabidussi. Our result also holds for the relation determining the prime decomposition of infinite connected graphs with respect to the weak Cartesian product. Moreover, we show that this relation is the transitive closure of the union of the starting relations of Sabidussi and Winkler [European J.Combin. 8 (1987), 209-212], thereby generalizing the result of Feder [J. Graph Theory, to appear] from finite to infinite graphs.

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1 Introduction

The Cartesian product is one of several standard products of graphs. As shown by Sabidussi [10] and later by Vizing [12] all finite connected graphs have unique prime factorizations. Recently two polynomial algorithms for finding these factorizations were published. Based on Sabidussi's method Feigenbaum, Herschberger and Schäffer [4] developed a decomposition algorithm with running time $O(n^{4.5})$, where n is the number of vertices of the graph. Independently, Winkler [13] presented an $O(n^4)$ algorithm which is based on a method of isometrically embedding graphs into Cartesian products by Graham and Winkler [5]. This algorithm was later improved by Hochstrasser [6] to run in $O(mn + n^2(\log n)^2)$ time, where m is the number of edges. It depended on an improvement of the isometric embedding algorithm of Graham and Winkler by Aurenhammer and Hagauer [2].

Results of Feder [3] imply that the product relation which defines the prime factorization, is the transitive closure of the union of the starting relations of the algorithms by Feigenbaum et al. and by Winkler. As Feder already remarks, this leads to a simple algorithm with running time $O(nm)$. Due to its simplicity, this algorithm has high inherent parallelism and is efficient on the P-RAM parallel model of computation. In other words, it has polylogarithmic time complexity on a polynomial number of processors.

It should be noted, however, that the best known algorithm for decomposing a connected graph into prime factors with respect to the Cartesian product is due to Aurenhammer, Hagauer and Imrich [1]. It has time complexity $O(m \log n)$ and is based on an entirely different approach. Unfortunately this method does not seem to be easily parallelizable.

Infinite connected graphs can have infinitely many prime factors with respect to the Cartesian product. In this case they cannot be the Cartesian product of these factors. They are, as has been conjectured by Sabidussi [10] and shown by Miller [9] and Imrich [7], a connected component of the product of these graphs, i.e. a so called weak Cartesian product. This representation is unique in the sense that it defines a unique equivalence relation σ on the edge set of the graph in question.

In this paper we prove that σ is the convex hull of δ and thereby give an alternate direct proof of the unique factorization theorem for connected graphs, finite or infinite. Furthermore, it is shown for the same class of graphs that the relation σ is the transitive closure of the union of the starting relations δ of the algorithm by Feigenbaum et al. and the relation Θ of Winkler. In symbols, $(\delta \cup \Theta)^\star = \sigma$.

2 Preliminaries

All graphs considered in this paper are finite or infinite undirected graphs without loops or multiple edges. If G is a graph, we shall write $V(G)$ or V for its vertex set and $E(G)$ or E for its edge set. $E(G)$ shall be considered as a set of unordered pairs $\{x, y\}$ of distinct vertices of G.

Considering G as $V(G) \cup E(G)$, we shall often write $x \in G$ for $x \in V(G)$ and $e \in G$ for $e \in E(G)$.

Let G_ι, $\iota \in I$ be a set of graphs. Then the *Cartesian product* $G = \Box_{\iota \in I} G_\iota$ is defined as follows:

(i) $V(G)$ is the Cartesian product of the vertex sets of the factors. In other words, $V(G)$ is the set of functions $x : \iota \mapsto x_\iota$, $x_\iota \in V(G_\iota)$ of I into $\cup_{i \in I} V(G_\iota)$.

(ii) $E(G)$ consists of all unordered pairs $\{x, y\}$ of distinct vertices of G to which there exists a $\kappa \in I$ such that $\{x_\kappa, y_\kappa\} \in E(G_\kappa)$ and $x_\iota = y_\iota$ for $\iota \in I \setminus \{\kappa\}$.

In the case of two factors, G, H we obtain the usual Cartesian product $G \Box H$. It is commutative and associative in an obvious way, having the trivial graph as a unit.

Common examples of Cartesian products are squares, the skeletons of cubes and n–cubes, prisms (Cartesian products of n–gons by an edge) or the square lattice as the product of two infinite paths.

The weak infinite dimensional cube whose vertices are all 0–1 sequencies with only finitely many ones and where two vertices are connected by an edge if they differ in exactly one coordinate is a weak Cartesian product of countably many edges. (Of course, all such products are isomorphic.)

The product of finitely many graphs is connected if and only if every factor is. However, a product of infinitely many nontrivial graphs must be disconnected because it contains vertices differing in infinitely many coordinates. Two such vertices can never be connected by a path of finite length, because every edge connects vertices differing in exactly one coordinate.

This gives rise to the notion of the so-called weak Cartesian product:

Let G_ι, $\iota \in I$, be a set of connected graphs and $a \in V(\Box_{\iota\in I} G_\iota)$. Then the *weak Cartesian product*

$$G = \Box^a_{\iota\in I} G_\iota$$

is the connected component of $G = \Box_{\iota\in I} G_\iota$ containing a. We note that $\Box^a G_\iota = \Box^b G_\iota$ if and only if a and b differ in at most finitely many coordinates.

Now we define two relations σ and δ on $E(G)$ and state several known properties of σ and δ.

Definition. Let $G = \Box^a_{\iota\in I} G_\iota$. Call $e = \{u, v\}$ a G_κ–edge if e connects vertices u and v such that $u_\iota = v_\iota$ for all $\iota \neq \kappa$ and $\{v_\kappa, u_\kappa\} \in E(G_\kappa)$. We say that two edges e, f are in the relation $\sigma(\Box^a_{\iota\in I} G_\iota)$ if there is an ι such that both e and f are G_ι edges.

It is easy to see that $\sigma(\Box^a_{\iota\in I} G_\iota)$ is an equivalence relation. We shall later show that there exists a finest relation induced by a Cartesian product representation of G. This relation will be denoted by σ_G or simply σ.

Definition. Let $e, f \in E(G)$. We say e and f are in the relation δ if one of the following two conditions is satisfied

(1) e and f are opposite edges of a chordless square.

(2) e and f are adjacent and there is no chordless square containing e and f or $e = f$.

Clearly δ is reflexive and symmetric and its transitive closure $\delta^\star$ is an equivalence relation.

We wish to remark that Sabidussi introduced several relations in his paper on Graph Multiplicaton [10]. δ is essentially his $\sim_0$, $\delta^\star$ is $\sim_1$ and σ is $\sim$.

By the definition of δ any pair of adjacent edges which belong to distinct δ equivalence classes span a square without a diagonal. Furthermore, if these edges are in different classes with respect to $\delta^\star$, then there is exactly one such square. We say that the relation $\delta^\star$ has the *square property*. It is easy to see that the square property also holds for any equivalence relation containing $\delta^\star$. Note that every product relation $\sigma(\Box^a_{\iota\in I} G_\iota)$ contains δ and thus has the square property.

Let H be a subgraph of G. Then H is *convex* (in G) if all shortest G–paths between two vertices of H are already in H.

The motivation for this work was to investigate the importance of convexity for the Cartesian product and the interdependence of the relations δ, Θ and σ. As an example, we point out that so-called layers and subproducts of a Cartesian product are convex.

In a more general context, Tardif uses the convexity of prefibres when proving the uniqueness of factorization of metric spaces [11].

A natural way for defining convexity of relations is to call a relation convex if the graphs induced by its equivalence classes are convex. It turns out, however, that such a definition is not strong enough for our purpose. Therefore we propose the following definition:

Definition. Let γ be an equivalence relation with equivalence classes E_ι, $\iota \in I$. We say γ is *convex* if for any $K \subseteq I$ every connected component of the graph induced on $\cup_{\iota \in K} E_\iota$ is convex.

The most important convex relations considered here are the product relation $\sigma(\Box^a_{\iota \in I} G_\iota)$ and Djoković' relation Θ.

It is not obvious from the above definition whether the intersection of convex relations is convex. We shall later prove this for equivalence relations containing δ (Lemma 4). Hence, for the relations of interest in this paper we can define the *convex hull* $\mathcal{C}(\rho)$ of a relation ρ to be the minimal convex equivalence relation containing ρ. More precisely, $\gamma = \mathcal{C}(\rho)$ if γ is convex, contains δ and ρ and if for any convex relation $\gamma' \supseteq \rho$ we have $\gamma \subseteq \gamma'$.

3 Results

Lemma 1 *Let γ be an equivalence relation on the edge set $E(G)$ of a connected graph. Suppose γ has the equivalence classes $\gamma_1, \gamma_2, \dots, \gamma_k, \dots$ and satisfies the square property. Then every vertex of G meets every γ_ι.*

Note that we used Greek letters for the equivalence classes of γ instead of the capital Roman letters we used before. This is useful to avoid confusion with factors of graphs, which are also denoted by indexed capital Roman letters.

Proof. Assume there is an equivalence class γ_ι, such that there are vertices which do not meet it. By connectivity, there must be a pair of adjacent vertices, one of which meets γ_ι (i.e. meets an edge of γ_ι), but not the other. The edge connecting the two vertices belongs to one of the equivalence classes $\gamma_\kappa \neq \gamma_\iota$. Now application of the square property leads to a contradiction. $\square$

Lemma 2 *Let γ be an equivalence relation on the edge set $E(G)$ of a connected graph G satisfying the square property and let τ be an equivalence class of γ. If all connected components of the subgraph induced by τ are convex, they are isomorphic.*

Proof. Let H be the subgraph induced by τ. It suffices to show that any two components H_1, H_2 of H which are connected by an edge are isomorphic.

Suppose $\{x, y\}$ is an edge of G connecting H_1 with H_2. In order to define an isomorphism $\varphi : H_1 \to H_2$ we choose a spanning tree T of H_1 and set $\varphi x = y$. Let $v \neq x$ be an arbitrary vertex of H_1 and let u be the neighbor of v in T that lies on the unique path from v to x in T. If φu is already defined, we let φv be the unique vertex in G that completes $\{v, u\}$ and $\{u, \varphi u\}$ to a square.

We thus obtain a mapping from $V(H_1)$ into $V(H_2)$. By the square property, adjacent vertices in H_1 have different images in $V(H_2)$. Nonadjacent vertices of $V(H_1)$ have different images by convexity. Hence, φ is injective. If φT does not span H_2, we extend it to a spanning tree S of H_2 and extend φ^{-1} to an injective mapping ψ from H_2 to H_1. Clearly ψ and φ^{-1} must be identical.

Clearly φ depends on T and isomorphically maps T onto φT. Suppose φ^* is defined by a spanning tree T^* of H_1. By the square property and convexity it must coincide with φ. Since every edge of H_1 (and H_2) is in some spanning tree, this implies that φ is an isomorphism between H_1 and H_2. Note that $\{u, \varphi u\} \in E(G)$ for every $u \in H_1$ and that φu is the only neighbor of u in H_2. $\square$

Lemma 3 *Let γ be an equivalence relation on the edge set $E(G)$ of a connected graph G. Suppose γ satisfies the square property and has only two equivalence classes ρ, τ. Let H and K be the subgraphs induced by ρ and*

τ, with connected components $H_1, H_2, \ldots, H_r, \ldots$ and $K_1, K_2, \ldots, K_s, \ldots$, respectively. Then

$$|H_i \cap K_j| \geq 1.$$

If H_i and K_j are convex, we even have $|H_i \cap K_j| = 1$.

Proof. Suppose there is a pair H_i, K_j of disjoint components. W.l.o.g. we can assume that they have minimal distance. Let P be a shortest path from H_i to K_j. Clearly the first edge, say $\{x, y\}$ of P must be in τ and y is not in H_i. Suppose it is in H_k. Since the distance from H_k to K_j is smaller than the one from H_i to K_j

$$H_k \cap K_j \neq \emptyset.$$

Suppose $v \in H_k \cap K_j$ and let Q be a path from y to v in H_k. By repeated application of the square property we obtain a vertex u in H_i connected to v by an edge in τ. But then u is in K_j and $|H_i \cap K_j| \geq 1$. The fact that the intersection cannot contain more than one vertex if H_i and K_j are convex is obvious. $\Box$

Theorem 1 *Let γ be a convex equivalence relation on the edge set $E(G)$ of a connected graph G. If γ satisfies the square property it induces a factorization of G with respect to the weak Cartesian product.*

Proof. Suppose first that γ has only two equivalence classes ρ, τ. Define $H_1, H_2, \ldots, H_r$ and $K_1, K_2, \ldots, K_s$ as in Lemma 3. Let v be a vertex of G, H_i be the component of H containing v and K_j be the component of K containing v. Clearly, (j, i) is uniquely determined by v. We call (j, i) the coordinates of v and assign such coordinates to every vertex of G. Since $H_i \cap K_j = 1$ for every pair H_i, K_j, different vertices must have different coordinates and every pair (j, i), $1 \leq i \leq r$, $1 \leq j \leq s$, is assigned to some vertex v.

Let H_i and H_k be given. If they are connected by an edge the mapping

$$(j, i) \mapsto (j, k), \qquad 1 \leq i \leq r$$

clearly is an isomorphism from H_i onto H_k by Lemma 2. If H_i and H_k are connected by a path, this mapping is seen to be an isomorphism by induction on the length of the path.

An analogous observation holds for the components of K.

Thus, two vertices (j,i), (l,k) of G are connected if and only if they are in the same H_i or in the same K_j. If they are in the same H_i, we have $i=k$ and $\{(j,1),(l,1)\}\in E(H_1)$ by the isomorphism property. If these vertices are in K_j we have $j=l$ and $\{(1,i),(1,k)\}\in E(K_1)$. In other words, G is isomorphic to $H_1\Box K_1$.

In the sequel it will be convenient to define graphs G_1 and G_2 isomorphic to H_1 and K_1, respectively, by setting $V(G_1)=\{i\mid (i,1)\in V(H_1)\}$ and $V(G_2)=\{j\mid (1,j)\in V(K_1)\}$ such that the mappings $i\mapsto (i,1)$ and $j\mapsto (1,j)$ are isomorphisms. Hence, $G=G_1\Box G_2$.

Now assume there are arbitrarily many equivalence classes γ_ι, $\iota\in I$, of γ. For γ_ι define $\tau_\iota=\cup_{\kappa\neq\iota,\kappa\in I}\gamma_\kappa$. Let R_ι be relation with only two equivalence classes: γ_ι and τ_ι. By the above this gives a factorization of G into two factors $G_\iota\Box K$, where G_ι corresponds to γ_ι and K to τ_ι. By the ι–th coordinate v_ι of a vertex $v\in G$ we then mean the G_ι–coordinate of v in $G_\iota\Box K$.

From this definition it is clear that each vertex is assigned coordinates. If u and v have the same ι-th coordinate, this implies (by convexity) that there can be no edge of the equivalence class γ_ι on any shortest path between u and v. Now, if u and v have equal coordinates, there can be no nontrivial shortest path between them. Hence, if u and v have the same coordinates, then $u=v$. The assignment of coordinates to vertices of a connected graph G is thus bijective.

Two vertices u,v in G are adjacent iff they differ in exactly one coordinate, say the ι–th, and if u_ι, v_ι are adjacent in G_ι. Moreover, u and v can differ in at most finitely many coordinates since G is connected. Thus $G\cong\Box^a_{\iota\in I}G_\iota$ for any $a\in V(G)$. $\Box$

The equivalence relation on $E(G)$ whose only equivalence class is $E(G)$ itself is clearly convex and trivially satisfies the square property. Hence, in any graph there is at least one convex relation satisfying the square property. Any such relation gives a factorization by Theorem 1. The idea now is to take the the intersection of all such relations to get the prime factorization. To this end we first have to check, whether the intersection of an arbitrary set of convex relations with the square property is convex and has the square property.

There is no problem with the square property. Let γ_i, $i\in I$ be the set of all relations on G satisfying the square property. Then $\cap_{i\in I}\gamma_i$ clearly has the square property.

For the intersection of convex relations we prove the following Lemma.

Lemma 4 *Let γ_j, $j \in J$ be an arbitrary set of convex relations containing δ on G. Then $\gamma = \cap_{j\in J}\gamma_j$ is convex.*

Proof. Let u, v be any pair of vertices of G. Let E_ι, $\iota \in I$ be the equivalence classes of γ. First we shall prove that if P and Q are two shortest paths from u to v in G, then $x_\iota = y_\iota, \forall \iota \in I$ where $x_\iota = |P \cap E_\iota|$ and $y_\iota = |Q \cap E_\iota|$. Then we shall show that this implies the convexity of γ.

If the Lemma does not hold, then there is a graph with a pair of vertices, such that there are two shortest paths between them which provide a counterexample. These two paths meet a finite number of equivalence classes E_i, i.e. there are finitely many E_i with $x_i \neq 0$ or $y_i \neq 0$. There is at least one such example, where this number is minimal. Assume k equivalence classes of γ are needed for the smallest counterexample. Thus we have

$$\begin{aligned} |P| &= x_1 + x_2 + \ldots + x_k \\ |Q| &= y_1 + y_2 + \ldots + y_k. \end{aligned}$$

W.l.o.g. we can assume $x_1 \neq y_1$. Since $E_1 \neq E_2$ there is a γ_j such that E_1 and E_2 are subsets of different equivalence classes of γ_j. Let I_1 be the index set of eqivalence classes of γ, such that $F_1 = \cup_{i\in I_1} E_i$ is an equivalence class of γ_j and $E_1 \subseteq F_1$. The set $F_2 = \cup_{i\in I_2} E_i$, $I_2 = I \setminus I_1$ is then both a union of γ equivalence classes and a union of γ_j equivalence classes. By assumption we have

$$\sum_{i\in I_1} x_i + \sum_{i\in I_2} x_i = \sum_{i\in I_1} y_i + \sum_{i\in I_2} y_i.$$

By the square property of γ_j the paths P and Q have corresponding paths P', Q' with $|P' \cap E_i| = |P \cap E_i|$, $|Q' \cap E_i| = |Q \cap E_i|$ where each of them can be divided into two subpaths $P' = P_1 + P_2$, $Q' = Q_1 + Q_2$ such that P_1 and Q_1 are F_1-paths and P_2 and Q_2 are F_2-paths, i.e. $P_1 \cap F_2 = \emptyset$, $Q_1 \cap F_2 = \emptyset$ $P_2 \cap F_1 = \emptyset$, $Q_2 \cap F_1 = \emptyset$. Since F_1 and F_2 are unions of equivalence classes of a convex relation, it is easy to see that the endpoints of P_1 and P_2 (different from u) are identical. For, if the endpoints, say x and y, are different, there is a nontrivial shortest path from x to y. But, by convexity, this path has to be in F_1 and at the same time in F_2, which is not possible. Hence either the paths P_1 and Q_1 or the paths P_2 and Q_2 provide a counterexample with $k' < k$. This contradicts the minimality of k.

Thus we proved that for any two shortest paths from u to v in G, $|P \cap E_\iota| = |Q \cap E_\iota|$ for all E_ι. We have to show that this implies the convexity of γ.

Assume that γ is not convex. Then there must be unions of eqivalence classes of γ, say F_1 and $F_2 = E \setminus F_1$, and a pair of vertices, say u and v, such that there is a shortest path P from u to v contained in F_2, i.e. $P \cap F_1 = \emptyset$, and a shortest path Q from u to v intersecting F_1. Hence there is an edge $e \in Q \cap F_1$. Let $E_\iota \subset F_1$ be the equivalence class of e. We have

$$|Q \cap E_\iota| \neq |P \cap E_\iota| = 0.$$

Hence if $|P \cap E_\iota| = |Q \cap E_\iota|$ for all equivalence classes E_ι then γ must be convex. This completes the proof. □

The last Lemma implies that there is exactly one finest convex relation with the square property and this relation is exactly the intersection of all convex relations containing δ, i.e. the convex hull $\mathcal{C}(\delta)$. It is easy to see that this relation corresponds to a prime factorization of G and that this factorization is unique. We have thus proved the following two results:

Theorem 2 (Miller[9],Imrich[7]) *Every connected graph has a unique representation as a weak Cartesian product.*

Theorem 3 *The relation corresponding to the unique prime factorization of a connected graph G is the convex hull of δ. In symbols, $\sigma = \mathcal{C}(\delta)$.*

For a further consequence of the above statements we define Djoković' relation Θ.

Definition. Let $e = \{x, y\} \in E(G)$ and $f = \{x', y'\} \in E(G)$ be two edges of G. We say that e and f are in the relation Θ if $d(x, x') + d(y, y') \neq d(x, y') + d(x', y)$.

It was shown in [5] for finite graphs and in [8] for infinite ones, that any two shortest paths P and Q between two vertices u, v in a connected graph G

meet the equivalence classes of $\Theta^\star(G)$ equally often, i.e. for any equivalence class E_ι of $\Theta^\star$ we have

$$|P \cap E_\iota| = |Q \cap E_\iota|.$$

By the arguments in the proof of Lemma 4 this implies that $\Theta^\star$ is convex (end hence every equivalence relation containing it.) It is not hard to see that $\Theta \subseteq \sigma$. Consider $\rho = (\delta \cup \Theta)^\star$. Since σ is transitive, $\rho \subseteq \sigma$. On the other hand, ρ has the square property because it contains $\delta^\star$ and is convex because it contains $\Theta^\star$, hence $\sigma \subseteq \rho$ and we have $\rho = \sigma$. Thus:

Theorem 4 *Let G be a connected, infinite graph. Then $\sigma = (\delta \cup \Theta)^\star$.*

Recently a stronger analogue of this result was independently shown by Feder for finite products [3]. Instead of δ Feder uses a relation τ, defined as follows: Two edges $e = \{x, z\}$ and $f = \{z, y\}$ satisfy $e\tau f$ if z is the unique common neighbor of x and y.

Lemma 5 *Let γ be a convex equivalence relation containing τ. Then γ contains δ.*

Proof. Let e and f be two adjacent edges. If there is no square containing e and f then both $e\tau f$ and $e\delta f$ follow directly from definitions.

Assume now that e and f are contained in a square and let e' and f' be the their opposite edges. By definition of δ, $e\delta e'$ and $f\delta f'$. Hence, we wish to show that $e\gamma e'$ and $f\gamma f'$. This follows from convexity of γ by the following arguments. Assume e and e' are not in the same equivalence class of γ. If e is the only edge of its equivalence class, say E_1 on the square $\{e, f, e', f'\}$, this contradicts convexity of $E \setminus E_1$. If e and at least one of the edges f, f' are in the same class E_1, then by convexity, the whole square is in E_1, which is not possible. Thus $e\gamma e'$ and similarly for $f\gamma f'$. $\Box$

Theorem 5 *Let G be a connected, infinite graph. Then $\sigma = (\tau \cup \Theta)^\star = \mathcal{C}(\tau)$.*

Proof. Clear.

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