SEPARATING DOUBLE RAYS IN LOCALLY FINITE PLANAR GRAPHS

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Abstract

The main aim of this paper is to characterize infinite, locally finite, planar, 1-ended graphs by means of path separation properties. Let Γ be an infinite graph, let Π be a double ray in Γ, and let d and d_Π denote the distance functions in Γ and in Π, respectively.

One calls Π a *quasi-axis* if $\liminf d(x,y)/d_\Pi(x,y) > 0$, where x and y are vertices of Π and $d_\Pi(x,y) \to \infty$. An infinite, locally finite, almost 4-connected, almost-transitive, 1-ended graph is shown to be planar if and only if the complement of every quasi-axis has exactly two infinite components. Let Γ be locally finite, planar, 3-connected, almost-transitive, and 1-ended.

It is shown that no proper planar embedding of Γ has an infinite face and hence its covalences are bounded. If Γ has bounded covalences and if Π is any double ray in Γ, it is shown that $\Gamma - \Pi$ has at most two infinite components, at most

one on each side of Π. If, moreover, Π is a quasi-axis, then $\Gamma - \Pi$ is shown to have exactly two infinite components. With the aid of a result of C. Thomassen, the above-stated characterization of infinite, locally finite, planar, 1-ended graphs is then obtained.

1. Preliminaries

The graphs considered in this article are simple graphs. Capital Greek

letters will be reserved for graphs and their subgraphs. The symbols $V(\Gamma)$, $E(\Gamma)$, $\mathrm{AUT}(\Gamma)$, and $\kappa(\Gamma)$ will denote, respectively, the vertex set, the edge set, the automorphism group, and the connectivity of Γ. If $x \in V(\Gamma)$, then $\rho(x)$ will denote the valence of x. If $\rho(x) = r$ for all $x \in V(\Gamma)$, that is, if ρ is constant on $V(\Gamma)$, then we say that Γ is *ρ-valent* and write $\rho(\Gamma) = r$. In this article all infinite graphs will be presumed to be *locally finite*; that is, $\rho(x) < \infty$ for every vertex x. If $x, y \in V(\Gamma)$, then $d(x,y)$ denotes the distance between x and y. If Σ is a path or ray or double ray in Γ and if $x, y \in V(\Sigma)$, then $\Sigma[x,y]$ will denote the subpath, or segment, of Σ joining x and y. The meanings of $\Sigma(x,y)$, $\Sigma[x,\infty)$, etc., are then obvious. If Λ is a subgraph of Γ (denoted: $\Lambda \subseteq \Gamma$), then $\partial(\Lambda)$ denotes the set of those vertices not in $V(\Lambda)$ that are adjacent to a vertex of Λ.

We say that Γ is *almost-transitive* if $\mathrm{AUT}(\Gamma)$ acts on $V(\Gamma)$ with finitely many orbits. In a locally finite graph, there will, of course, be finitely many orbits in $E(\Gamma)$ as well. This notion generalizes the familiar notions of vertex-transitivity and edge-transitivity.

Suppose that Γ is connected. The cardinality of a smallest separating set $T \subset V(\Gamma)$ such that $\Gamma - T$ has at least one finite component is denoted by $\kappa_f(\Gamma)$, and such a set T, if it is finite, is called a *κ_f-separator.*

A finite subgraph Φ of Γ is an *atom* of Γ if Φ has the least number of vertices of any component of $\Gamma - T$ as T ranges over the κ_f-separators of Γ. We denote by $\alpha(\Gamma)$ the number of vertices in an atom of Γ. The cardinality of a smallest set $T \subset V(\Gamma)$ such that $\Gamma - T$ has at least two infinite components is denoted by $\kappa_\infty(\Gamma)$.

Each of the parameters κ_f and κ_∞ is defined to be ∞ in the event that an appropriate "smallest set" fails to exist. In any case, $\kappa(\Gamma) = \min\{\kappa_f(\Gamma), \kappa_\infty(\Gamma)\}$. We say that Γ is *almost 4-connected* if $\kappa_f(\Gamma) \geq 3$ and for every κ_f-separator T, $\Gamma - T$ has exactly two components, one consisting of a single vertex.

We use the notion of an "end" as formulated by Halin [1]. The number of ends of an infinite, locally finite graph Γ turns out to be the supremum, if it exists, of the number of infinite components of $\Gamma - T$ as T ranges over all finite subsets of $V(\Gamma)$; if this supremum does not exist, we say simply that Γ is *∞-ended.* When, moreover, Γ is almost-transitive, this supremum is 1, 2, or infinity (by [3, Corollary 15] combined with [5, Theorem 1]). Since this article is concerned mainly with 1-ended graphs, it is handy to note that the following (adapting [1,¶1.1]) are equivalent:

(1) Γ has exactly one end;

(2) For any finite subset T of $V(\Gamma)$, $\Gamma - T$ has just one infinite component, i.e., $\kappa_\infty(\Gamma) = \infty$;

(3) Any two rays in Γ are joined by an infinite sequence of pairwise-disjoint paths.

It follows that a 1-ended, almost 4-connected graph is 3-connected.

When a planar graph is 3-connected, the cyclic order of the edges incident with each vertex when that graph is embedded in the plane becomes an intrinsic property of the graph and is the same for all planar embeddings. This classic result of H. Whitney [11] for finite graphs has been generalized to infinite graphs in [4] and [8]. Also independent of the embedding is the collection of subsets of the edge-set that form the boundaries of the *faces*, i.e., the connected components of the complement of the embedded graph. Thus the *covalence* function ρ^* is a well-defined function on the set of faces of a planar, 3-connected graph, counting the number of edges incident with each face. That number may be infinite for some faces. If $\rho^*(F) = \infty$, then F will be called an *infinite face*.

For a biconnected planar graph, let us define a *proper embedding* to be a planar embedding with the property that the boundary of each face is either an elementary circuit or a double ray.

Halin [2] has characterized the forbidden subgraphs of a locally finite, planar graph that force all its planar embeddings to have at least one point of accumulation. All these forbidden subgraphs entail a finite subgraph that separates two infinite subgraphs. It follows that every 1-ended, locally finite, planar graph has an embedding without accumulation points.

Our reason for this concern is that we will be dealing with infinite faces, in particular, showing when they do *not* occur. Certain embeddings of an infinite, planar graph into the Euclidean or hyperbolic plane (for example, into a bounded subspace) not only force the existence of accumulation points, but may also affect the number of exterior infinite faces, so that this number would be dependent upon the planar embedding rather upon the graph itself. For example, a proper embedding of a graph consisting of a two-way infinite ladder has two infinite faces rather than just one. Every embedding without accumulation points (except perhaps at infinity) of a biconnected graph is proper. Therefore, *It will be understood that only proper embeddings of 1-ended graphs are used.*

2. Some Properties of 1-ended Graphs

In this section we obtain some fundamental results concerning the valence, connectivity, and covalence parameters in locally finite, 1-ended graphs.

Lemma 2.1. *If Γ is 1-ended and vertex-transitive, then $\rho(\Gamma) \geq 3$.*

Proof. Since Γ has exactly one end and is vertex-transitive, Γ is connected. Hence $\rho(\Gamma) \geq 2$. But if $\rho(\Gamma) = 2$, then Γ would be a double ray, which is 2-ended. □

Lemma 2.2. *If Γ is 1-ended and vertex-transitive, then $\kappa(\Gamma) \geq 3$.*

Proof. If $\kappa(\Gamma) < 3$, then $\kappa_f(\Gamma) < 3$ since Γ is 1-ended. By Lemma 2.1, $\kappa_f(\Gamma) < \rho(\Gamma)$, and so Γ has nontrivial atoms, that is, $\alpha(\Gamma) \geq 2$. But by [6, Corollary 3A], $2\alpha(\Gamma) \leq \kappa_f(\Gamma) \leq 2$, giving a contradiction. □

At this point planarity is added to our list of hypotheses.

Theorem 2.3. *Let Γ be locally finite, planar, 2-connected, and almost-transitive. If Γ is 1-ended, then it has no infinite faces (in any proper embedding).*

Proof. Suppose the double ray Π is the boundary of an infinite face F of Γ. Let $E(\Pi) = \{\{x_{i-1}, x_i\} : i \in \mathbb{Z}\}$ where (without loss of generality) x_{i+1} is the clockwise successor of x_{i-1} about the vertex x_i. Since the two rays $\Pi^+ = \Pi[x_1, \infty)$ and $\Pi^- = \Pi(-\infty, x_{-1}]$ must belong to the same unique end, there exists an infinite sequence of pairwise-disjoint (finite) paths $\{\Phi_i[x_{-m_i}, x_{n_i}]\}_{i=1}^{\infty}$, where $\{m_i\}_{i=1}^{\infty}$ and $\{n_i\}_{i=1}^{\infty}$ are increasing sequences of positive integers.

Case 1: F is the only infinite face of Γ. Since Π is invariant under AUT(Γ), then by almost-transitivity, there exists an integer k such that for all vertices x we have $d(x, V(\Pi)) < k$. We may assume that $k \geq 3$. Using the sequence $\{\Phi_i\}_{i=1}^{\infty}$, we will construct another such sequence such that the length of each path is $< 2k$.

Let $\Psi_1 = \Pi[x_{-1}, x_1]$. Assume that for some $m > 1$ we have already constructed pairwise-disjoint $\Pi^+\Pi^-$-paths $\Psi_1, \Psi_2, \ldots, \Psi_{m-1}$, each of length less than $2k$. Clearly there is a positive integer j such that none of $\Psi_1, \Psi_2, \ldots, \Psi_{m-1}$ uses a vertex in $\Pi(-\infty, x_{-m_j}] \cup \Pi[x_{n_j}, \infty)$ or intersects a path in $\{\Phi_i\}_{i=j}^{\infty}$.

To construct Ψ_m we do the following. By planarity, for each vertex in Φ_{j+k}, there exists a path of length $< k$ from that vertex to Π, and any such path meets Φ_i only if $i > j$. Since both $x_{-m_{j+k}}$ and $x_{n_{j+k}}$ are adjacent to interior vertices of Φ_{j+k}, there exists an edge $e = \{u, v\}$ of Φ_{j+k} such that $d(u, y) < k$ and $d(v, z) < k$, where $y \in V(\Pi(x_{n_j}, \infty))$ and

$z \in V(\Pi(-\infty, x_{-m_j}))$. Let Σ and Σ' be shortest paths joining u to y and v to z, respectively. Then the walk $\Sigma \cup \{e\} \cup \Sigma'$ contains a path Ψ_m with the desired properties.

Proceeding inductively, we construct the required infinite sequence of paths $\{\Psi_i\}_{i=1}^{\infty}$.

Evidently, there exists an infinite subset $\mathcal{P}$ of $\{\Psi_i \mid i \in \mathbb{Z}^+\}$ such that all paths in $\mathcal{P}$ have the same length n. Note that AUT(Γ) fixes $V(\Pi)$ and acts with finitely many orbits on the set of all paths of length n in Γ.

Also, by local finiteness, there are only finitely many paths of length n terminating at any given vertex of Π. Hence some path $\Psi = \Psi[x_a, x_b] \in \mathcal{P}$ is the image under some $\gamma \in$ AUT(Γ) of some other path $\Psi' = \Psi'[x_c, x_d] \in \mathcal{P}$. It follows that also $\gamma(\Pi[x_a, x_b])$ must be the path $\Pi[x_c, x_d]$, which is impossible since $|E(\Pi[x_c, x_d])| = |c - d| \neq |a - b| = |E(\Pi[x_a, x_b])|$. Hence this case is impossible.

Case 2: there exists an infinite face $F' \neq F$. Let Π' be the boundary of F' and let Ψ be a path joining Π and Π'. By reindexing if necessary, we may assume that Ψ runs from x_0 to some vertex on Π'. Since only finitely many of the paths Φ_i can meet Ψ, we select a path Φ_k that does not and consider the circuit Δ formed by Φ_k and the segment $\Pi[x_{-m_k}, x_{n_k}]$. But then Δ separates a cofinite subgraph of Π from a cofinite subgraph of Π', implying that Γ must have more than one end. This case, too, is impossible. $\square$

Corollary 2.4. *Let Γ be locally finite, planar, 3-connected, and almost-transitive. If Γ is 1-ended, then its set of covalences is bounded.*

Proof. It follows immediately from the hypotheses that Γ has finitely many face-orbits. Hence its dual graph Γ^* is almost-transitive, and by Theorem 2.3, Γ^* is locally finite. Hence Γ^* has bounded valences. $\square$

3. Separating Properties of Double Rays

For the sequence of lemmas in this section we assume that Γ is 1-ended, planar, and 3-connected and that Γ has been properly embedded in the plane. Let Π be a double ray in Γ. Continuing the notation of the previous section, we have

$$V(\Pi) = \{x_i \mid i \in \mathbb{Z}\}; \quad E(\Pi) = \{\{x_{i-1}, x_i\} \mid i \in \mathbb{Z}\}.$$

A neighbor y of x_i is a *left neighbor* (respectively, *right neighbor* of x_i) if $y \notin V(\Pi)$ and in clockwise (respectively, counterclockwise) order about x_i, one encounters y after x_{i-1} and before x_{i+1}. Let L (respectively, R) denote the set of vertices that share a component of $\Gamma - \Pi$ with a left (respectively, right) neighbor of a vertex of Π.

Lemma 3.1. *Every LR-path meets* Π.

Proof. If there exists an LR-path in $\Gamma - \Pi$, then by the definitions there exists such a path Σ whose terminal vertices are a left neighbor y of some x_i and a right neighbor z of some x_j. We may assume without loss of generality that $i \leq j$. It is then straightforward to show that the circuit $\Pi[x_i, x_j] \cup \{x_j, z\} \cup \Sigma \cup \{y, x_i\}$ separates the two connected subgraphs induced by $\{x_k \mid k < i\}$ and $\{x_k \mid k > j\}$, contrary to the assumption that Γ is 1-ended. $\square$

A trivial consequence of Lemma 3.1 is that $L \cap R = \emptyset$. An edge of Γ is said to be *on the left side* of Π if it is incident with a vertex in L. A face of Γ is said to be *on the left side* of Π if it is incident with an edge that is on the left side. The term *on the right side* is defined analogously. Note that by an argument similar to the proof of Lemma 3.1 (and by the definition of a proper embedding), no face can be on both sides of Π.

We denote by $\mathcal{L}$ (respectively, $\mathcal{R}$) the set of components of $\Gamma - \Pi$ whose vertex set is contained in L (respectively, in R). Since Γ is connected, every component of $\Gamma - \Pi$ belongs to exactly one of $\mathcal{L}$ and $\mathcal{R}$ and is called a *left component* or a *right component* accordingly.

Lemma 3.2. *Let Λ_1 and Λ_2 be distinct elements of $\mathcal{L}$. Suppose that for some integers $i < j < k$ one has $x_i, x_k \in \partial(\Lambda_1)$ and $x_j \in \partial(\Lambda_2)$. Then*

(1) $\partial(\Lambda_2) \subseteq \{x_m \mid i \leq m \leq k\}$;

(2) Λ_2 is a finite component.

Proof. (1) Clearly $\partial(\Lambda_2) \subseteq V(\Pi)$. Let $x_m \in \partial(\Lambda_2)$. There exists an $x_i x_k$-path Φ_1 all of whose intermediate vertices are in Λ_1 and an $x_j x_m$-path Φ_2 all of whose intermediate vertices are in Λ_2. Since Φ_1 and Φ_2 have no common vertex and have all their edges on the same side of Π, all the edges of Φ_2 lie on the same side of the circuit $\Phi_1 \cup \Pi[x_i, x_k]$. Hence $i \leq m \leq k$.

(2) By part (1) $\partial(\Lambda_2)$ is finite. If Λ_2 were infinite, then Γ would have at least two ends. $\square$

An immediate consequence is the following:

Lemma 3.3. *Given any two distinct infinite elements of $\mathcal{L}$, they may be labeled Λ_1 and Λ_2 so that $i \leq j$ holds whenever $x_i \in \partial(\Lambda_1)$ and $x_j \in \partial(\Lambda_2)$. Furthermore, $|\partial(\Lambda_1) \cap \partial(\Lambda_2)| \leq 1$.*

It is clear by symmetry that one may substitute $\mathcal{R}$ for $\mathcal{L}$ in Lemmas 3.2 and 3.3.

Lemma 3.4. *If Γ has no infinite face, then $\mathcal{L}$ and $\mathcal{R}$ each include at most one infinite element.*

Proof. Suppose that $\mathcal{L}$ includes distinct infinite elements Λ_1 and Λ_2. By Lemma 3.3 we may assume that there exist $m \in \mathbb{Z}$ such that $m = \max\{i \mid x_i \in \partial(\Lambda_1)\}$.

We obtain a contradiction by constructing an infinite face F. Let $y_0 = x_m$. Proceeding counterclockwise about y_0 from x_{m+1}, let y_{-1} be the first vertex of

Λ_1 encountered. Continuing inductively in the negative sense, for each $i \in \mathbb{Z}^+$, proceeding counterclockwise about y_{-i} from y_{-i+1}, let y_{-i-1} be the first vertex encountered. Were it to hold that $y_{-i} = x_h$ for some $h < m$, then, since Λ_1 is infinite and connected, the finite set $\{y_{-1}, y_{-2}, \cdots, y_{-i} = x_h, x_{h+1}, \cdots, x_m = y_0\}$ would separate two infinite subgraphs of Γ. Hence $y_{-i} \in V(\Lambda_1)$ for all $i \in \mathbb{Z}^+$. Since the boundary of F contains a ray, F

must be an infinite face.

One may substitute $\mathcal{R}$ for $\mathcal{L}$ in the foregoing argument by substituting "clockwise" for "counterclockwise"

in the foregoing construction. $\square$

Theorem 3.5. *Let Γ be a locally finite, planar, 2-connected,*

1-ended, and almost-transitive. Then for any double ray Π in Γ, $\Gamma - \Pi$ has at most two infinite components, of which, for any proper planar embedding of Γ, at most one is a left component and at most one is a right component.

Proof. Apply Theorem 2.3 followed by Lemma 3.4. $\square$

Let T_r denote the (infinite) r-valent tree. When $r \geq 3$ the cartesian product $T_r \times T_2$ satisfies all the hypotheses of Theorem 3.5 except that it is nonplanar. Let Π be the double ray

(a copy of T_2) induced by any fiber of the projection $T_r \times T_2 \to T_r$.

Then $(T_r \times T_2) - \Pi$ consists of r infinite components, and we see that

planarity is an indispensible hypothesis for the theorem.

[Insert Figures 1a and 1b approximately here.]

The following example shows that 1-endedness is also an indispensible hypothesis. Begin with the 4-valent regular tessellation of the Euclidean plane, and let certain of its 4-gons be distinguished, as indicated by the $\times$ in Figure 1a. The same graph is shown in Figure 1b but embedded with one of the distinguished 4-gons as its exterior face. Let Γ be formed by first placing a copy of Figure 1b onto each of the distinguished 4-gons of Figure 1a, identifying the boundaries. Then onto each of the 4-gons distinguished by $\times$ of this new graph, again superimpose a copy of Figure 1b. Care must be taken at each stage to orient the copies consistently to preserve vertex-transitivity. Γ is the limit graph when this process is repeated indefinitely. Γ satisfies all the hypotheses of the theorem except that it is infinitely-ended. The heavy black line in Figure 1a indicates a portion of a meandering double ray Π;

observe that $\Gamma - \Pi$ can have any desired number of components, even infinitely many.

Let Ξ be a double ray in an infinite graph Γ, and label $V(\Xi) = \{x_i \mid i \in \mathbb{Z}\}$ in the obvious way as in the previous sections. We define

$$\sigma(\Xi) = \liminf_{|i-j|\to\infty} \frac{d(x_i, x_j)}{|i-j|}$$

and call $\sigma = \sigma(\Xi)$ the *straightness* of Ξ. Clearly $0 \le \sigma \le 1$. We call Ξ a *quasi-axis* if $\sigma > 0$. Following [10], Ξ is an *axis* if $d(x_i, x_j) = |i-j|$ for all $i, j \in \mathbb{Z}$. Thus $\sigma = 1$ for all axes, but the converse is not true. For example, if Ξ is obtained from an axis by replacing finitely many edges by paths joining their incident vertices, then $\sigma(\Xi) = 1$, but Ξ is not necessarily an axis.

We saw in the previous section that, under suitable conditions on Γ, the complement of any double ray Π has at most two infinite components. It is possible for $\Gamma - \Pi$ to have only one infinite component or none at all. In the extreme case, if Π is a Hamilton path, then its complement has no vertices at all. By restricting the double rays under consideration to quasi-axes and replacing the hypothesis that there be no infinite face by the sharper condition that the covalences be bounded, we force the existence of two infinite faces in their complements.

Theorem 4.1. *Let Γ be locally finite, planar, 3-connected, and 1-ended, with bounded covalences. For any quasi-axis Ξ in Γ, $\Gamma - \Xi$ has exactly two infinite components, one on each side of Ξ.*

Proof. Let Ξ^+ and Ξ^- denote the two rays in Ξ emanating from x_0 whose vertices are those with nonnegative and nonpositive subscripts, respectively.

Since Γ is 1-ended, there exists an infinite set of pairwise-disjoint paths joining Ξ^+ to Ξ^-. By Lemma 3.1, each of these paths lies entirely on one side of Ξ. If on one side of Ξ there are only finitely many such paths, then it is straightforward (by the technique in the proof of Lemma 3.4) to construct an infinite face, contrary to hypothesis. Hence there are infinitely many of these paths on each side of Ξ.

If some two of these paths on the same side of Ξ are an $x_i x_j$-path with $i < j$ and an $x_m x_n$-path with $m < n$, then, as we saw in the proof of Lemma 3.2, one cannot have $i < m < j < n$. Hence there exists on the right side of Ξ a sequence of pairwise-disjoint paths $\{\Phi_k[x_{-m_k}, x_{n_k}]\}_{k=1}^{\infty}$ where $\{m_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ are increasing sequences of positive integers.

Suppose that no right component of Ξ is infinite. (The possibility that there are no vertices on the right side of Ξ is not hereby precluded.) Then there exists a subsequence $\{\Phi'_k\}$ of $\{\Phi_k\}$ such that no two distinct paths in the subsequence meet the same component of $\Gamma - \Xi$. For each $k \in \mathbb{Z}^+$, let x'_{-m_k} and x'_{n_k} denote the two terminal vertices of Φ'_k. For each $k \in \mathbb{Z}^+$, there exists a face F_k on the right side of Ξ on whose boundary lie vertices x_{-p_k} and x_{q_k} where $m'_k \leq p_k < m'_{k+1}$ and $n'_k \leq q_k < n'_{k+1}$, giving $\rho^*(F_k) \geq 2d(x_{-p_k}, x_{q_k})$.

¿From this and the hypothesis that Ξ is a quasi-axis, we have

$$\liminf_{k\to\infty} \frac{\rho^*(F_k)}{p_k + q_k} \geq 2\sigma > 0.$$

But then $\lim_{k\to\infty} \rho^*(F_k) = \infty$, contrary to hypothesis. Hence some right component of $\Gamma - \Xi$ is infinite. By symmetry there also exists an infinite left component of $\Gamma - \Xi$. By Lemma 3.1 these components are distinct. The theorem now follows from Lemma 3.4. □

Corollary 4.2. *Let Γ be locally finite, planar, 3-connected, 1-ended, and almost-transitive. If Ξ is a quasi-axis in Γ, then $\Gamma - \Xi$ has exactly two infinite components, one on each side of Ξ.*

Proof. Apply Corollary 2.4 and Theorem 4.1. □

Corollary 4.3. *Let Γ be locally finite, planar, 1-ended, and vertex-transitive. If Ξ is an axis in Γ, then $\Gamma - \Xi$ consists of exactly two components, both infinite, one on each side of Ξ.*

Proof. By [10, Theorem 4.2] $\Gamma - \Xi$ has no finite components. Now apply Lemma 2.2 and Corollary 4.2. □

Remark. Let Π be a double ray with $V(\Pi) = \{x_i \mid i \in \mathbb{Z}\}$ as in the previous sections. We call Π a *receding path* if

$$\lim_{j\to\infty} d(x_{i-j}, x_{i+j}) = \infty, \quad \text{for all } i \in \mathbb{Z}.$$

All quasi-axes clearly satisfy this condition. To see that not every receding path is a quasi-axis, we show by example that a receding path in graph Γ (where Γ satisfies the hypothesis of Theorem 4.1) need not have the property that its complement has an infinite component on each side. Suppose that Π together with its left side form a subgraph of Γ that is locally finite, planar, 3-connected, and 1-ended. Suppose that Π is an axis in this subgraph and that the set of covalences of the faces on the left side is bounded.

We now describe what happens on the right side of Π. (See Figure 2.) Let h be a fixed integer, $h \geq 2$. For each $k \in \mathbb{Z}^+$, let Φ_k and Ψ_k be paths of length h on the right side of Π that are disjoint except for their common terminal vertices x_{-k} and x_{k^2}. Suppose, moreover, that Φ_k is disjoint from Φ_m and Ψ_m when $m \neq k$ and that Ψ_k lies between Φ_k and Φ_{k+1}. Finally, let the regions between Ψ_k and Φ_{k+1} $(k \in \mathbb{Z}^+)$ and between Π and Φ_1 be triangulated by sufficiently many additional vertices and edges so that $d(x_{-k}, x_m) = h + (m - k^2)$ whenever $k^2 \leq m \leq k^2 + k + 1$. For each k, the face bounded by $\Phi_k \cup \Psi_k$ has covalence $2h$, and so Γ satisfies the hypothesis of Theorem 4.1.

[Insert FIGURE 2 approximately here.]

One easily verifies that for all $i \in \mathbb{Z}$,

$$d(x_{i-j}, x_{i+j}) = O(j - \sqrt{j})$$

and so Π is a receding path. The components of $\Gamma - \Pi$ on the right side of Π lying between and including Ψ_k and Φ_{k+1} form an infinite sequence of finite right components, but there is no infinite right component.

We now develop the machinery needed for our last theorem. Most of these ideas appeared originally in work of Thomassen [7, 9], but are discussed again here in order to make this paper as self-contained as possible.

Let the graph Γ be locally finite, nonplanar, 3-connected, almost 4-connected, and almost-transitive. Let Π be a double ray in Γ, and label $V(\Pi) = \{x_i \mid i \in \mathbb{Z}\}$ in the obvious way as in the previous sections. By [9, Lemma 4.3] for each $i \in \mathbb{Z}$, there exists a 2-connected, finite, nonplanar subgraph Δ_{x_i} that contains x_i. By [9, Lemma 4.5] there exists a sequence

$\{\Theta_i\}_{i=1}^{\infty}$ of subgraphs such that for each $i \in \mathbb{Z}^+$,

$\Theta_i \in \{\Delta_{x_1}, \Delta_{x_2}, \ldots\}$ and $\Theta_i \cap \Theta_j \neq \emptyset$

if and only if $|i - j| \leq 1$.

For each $i \in \mathbb{Z}^+$ and for each 2-path $\{\{u, v\}, \{v, w\}\}$ where

$u, v \in V(\Theta_i)$, $\Gamma - v$ contains a uw-path. Let Υ_i be the union of these uw-paths, taken over all 2-paths $\{\{u,v\},\{v,w\}\}$ where $u, v \in V(\Theta_i)$. Evidently the subgraph $\bigcup_{i=1}^{\infty}(\Theta_i \cup \Upsilon_i)$ is 2-connected.

For each $i \in \mathbb{Z}^+$ define a finite subgraph Ω_i of Γ as follows. For each 4-tuple u_1, u_2, u_3, u_4 of distinct vertices in Γ such that $\{u_1, u_2\}, \{u_3, u_4\} \in E(\Gamma)$ and $u_1, u_3 \in V(\Theta_i \cup \Upsilon_i)$, by [9, Lemma 4.4] there exist four internally disjoint paths from $\{u_1, u_2\}$ to $\{u_3, u_4\}$ such that each u_i for $1 \le i \le 4$ is a terminal vertex of precisely two of these paths. Ω_i denotes the union (taken over all the 4-tuples (u_1, u_2, u_3, u_4) defined above) of these four paths.

By [9, Lemma 4.6] it is possible to choose the subgraphs defined above so that $\Theta_i \cup \Upsilon_i \cup \Omega_i$ is 2-connected, finite and has diameter $\le k$, where k is some fixed integer. This fact immediately implies the following result.

Lemma 5.1. [9, Lemma 4.7] *For each finite subgraph Λ of Γ there exist $q_1, q_2, q_3 \in \mathbb{Z}$, such that for all integers q_4 there exist $q_5, q_6 \in \mathbb{Z}$ such that:*

(1) $\Lambda \cap (\Theta_i \cup \Upsilon_i \cup \Omega_i) = \emptyset$ *for* $i \ge q_1$

(2) $\Theta_j \cap \Upsilon_i = \emptyset$ *for* $j \le q_1, i \ge q_2$

(3) $\Theta_j \cap \Omega_i = \emptyset$ *for* $j \le q_2, i \ge q_3$

(4) $\Theta_j \cap \Omega_i = \emptyset$ *for* $j \ge q_5, i \le q_4$

(5) $\Theta_j \cap \Upsilon_i = \emptyset$ *for* $j \ge q_6, i \le q_5$.

If u_1, u_2, v_1, v_2 are distinct vertices of Γ, then a (u_1, u_2, v_1, v_2)-*linkage* consists of two disjoint paths $\Phi_i = \Phi_i[u_i, v_i]$ in Γ for $i = 1, 2$.

Lemma 5.2. [9, Lemma 4.8] *Let $q_1, q_2, \ldots, q_6 \in \mathbb{Z}$ such that $q_1 < \ldots < q_6$ and such that* (2), ... ,(5) *in Lemma 5.1 hold. Suppose further that q_4 is so large that*

(6) $\Upsilon_i \cap \Upsilon_q = \emptyset$ *if* $i \le q_3, q \ge q_4$.

Let $u_1 \in V(\Theta_{q_1}), v_1 \in V(\Theta_{q_6})$, and

$$u_2, v_2 \in \left[\bigcup_{i=q_3}^{q_4} V(\Omega_i)\right] \setminus \left[\bigcup_{i=1}^{\infty} V(\Theta_i) \cup V(\Upsilon_i)\right].$$

Then the subgraph Θ' of Γ induced by $V\left(\left[\bigcup_{i=q_1}^{q_6}\Theta_i\right] \cup \left[\bigcup_{i=q_2}^{q_5}\Upsilon_i\right] \cup \left[\bigcup_{i=q_3}^{q_4}\Omega_i\right]\right)$ *has an (u_1, u_2, v_1, v_2)-linkage.*

The proof of the following result is an adaptation of the proof of the main result in [9].

Theorem 5.3. *Let Γ be a locally finite, 1-ended, almost 4-connected, almost-transitive graph. Then Γ is planar if and only if for every quasi-axis Ξ, $\Gamma - \Xi$ has exactly two infinite components.*

Proof. The necessity follows from Corollary 4.2.

To prove sufficiency, let Π be any axis and let $\Theta_i, \Upsilon_i, \Omega_i$ for $i \in \mathbb{Z}^+$ be as above. It is an easy matter to adapt the proof of [11, Theorem 4.1] to show that an infinite, locally finite, almost-transitive graph indeed contains an axis. (In the above-cited theorem, the stronger hypothesis of vertex-transitivity was assumed.) Let Λ_1 and Λ_2 be the two infinite components of $\Gamma - \Pi$. We shall suppose Γ is nonplanar and alter Π slightly to construct a quasi-axis Ξ such that $\Gamma - \Xi$ has just one infinite component.

Let $q_1, q_2, q_3 \in \mathbb{Z}$ satisfy Lemma 5.1. Since Γ is almost-transitive and 1-ended, the subgraphs Λ_1' and Λ_2' induced by $V(\Pi \cup \Lambda_1)$ and $V(\Pi \cup \Lambda_2)$, respectively, are 1-ended. For $i = 1, 2$, there exists a $\Pi^+\Pi^-$-path Σ_i in Λ_i' that does not meet the finite subgraph $\bigcup_{i=1}^{q_3}(\Theta_i \cup \Upsilon_i \cup \Omega_i)$. Let y_i be the terminal vertex of Σ_i on Π^+. Some subpath Σ of $\Sigma_1 \cup \Pi[y_1, y_2] \cup \Sigma_2$ is a

$\Lambda_1\Lambda_2$-path that does not meet $\Theta_i \cup \Upsilon_i \cup \Omega_i$

when $i \leq q_3$ but meets Θ_j for some $j > q_3$. Then

$\Sigma \cap (\bigcup_{i=1}^{\infty}(\Theta_i \cup \Upsilon_i \cup \vee mega_i))$ is a

u_1v_1-path for some vertices u_1 and v_1. Since Ω_i includes all

vertices of Γ that have a neighbor in $\Theta_i \cup \Upsilon_i$, both

$u_1, v_1 \notin \bigcup_{i=1}^{\infty}(\Theta_i \cup \Upsilon_i)$. Choose q_4 such that $u_1, v_1 \in \bigcup_{i=q_3}^{q_4} \Omega_i$. Assume q_4 is so large that (6) in Lemma 5.2 holds. Choose q_5, q_6 such that Lemma 5.1 holds. Let u_2 and v_2 be the two vertices of $\Pi \cap \left(\bigcup_{i=q_1}^{q_6} \Theta_i\right)$ such that $\Pi \cap \left(\bigcup_{i=q_1}^{q_6} \Theta_i\right) \subseteq \Pi[u_2, v_2]$. Let Φ_2

be the u_2v_2-path in the (u_1, u_2, v_1, v_2)-linkage given by Lemma 5.2. Evidently $\Xi = (\Pi - \Pi[u_2, v_2]) \cup \Phi_2$ is the required quasi-axis. $\square$

Remark. Theorem 5.3 fails if Γ is assumed

merely to be 3-connected instead of almost 4-connected. To see this, consider the 6-valent planar triangular tessellation of the plane. For

every triangle adjoin a second triangle and all nine edges between the two. The resulting graph satisfies all the hypotheses of Theorem 5.3 except that it is not almost 4-connected. Furthermore, every quasi-axis separates the graph into two infinite components, yet it is nonplanar.

[Insert Figure 3 approximately here.]

Moreover, one may not replace "quasi-axis" by "axis" in Theorem 5.3, as the almost-transitive graph in Figure 3 demonstrates. All axes in this graph separate the graph into two infinite components, but again it is nonplanar. It is not known if "quasi-axis" can be replaced by "axis" for vertex-transitive graphs. Hence we offer the following conjecture.

Conjecture 5.4. *Let* Γ *be an infinite, locally finite,* 1*-ended, vertex-transitive graph. Then* Γ *is planar if and only if for every axis* Ξ*,* $\Gamma - \Xi$ *has exactly two components, both infinite.*

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[1] R. Halin, Über unendliche Wegen in Graphen, Math. Ann. 157 (1964) 125-137.

[2] R. Halin, Zur häufungspunktfreien Darstellung abzählbarer Graphen in der Ebene, Arch. Math. 17 (1966) 239-243.

[3] R. Halin, Automorphisms and endomorphisms of infinite locally finite graphs, Abh. Math. Sem. Univ. Hamburg 39 (1973) 251-283.

[4] W. Imrich, On Whitney's theorem on the unique imbeddability of 3-connected planar graphs, in: M. Fiedler, ed., Recent Advances in Graph Theory (Academia Praha, Prague, 1975) 303-306.

[5] H.A. Jung, A note on fragments of infinite graphs, Combinatorica 1 (1982) 285-288.

[6] H.A. Jung and M.E. Watkins, On the connectivities of finite and infinite graphs, Monatsh. Math. 83 (1977) 121-131.

[7] C. Thomassen, 2-linked graphs, Europ. J. Combin. 1 (1980) 371-378.

[8] C. Thomassen, Duality of infinite graphs, J. Combin. Theory (Ser. B) 33 (1982) 137-166.

[9] C. Thomassen, The Hadwiger number of infinite vertex-transitive graphs, Combinatorica 12 (1992) 481-491.

[10] M.E. Watkins, Infinite paths that contain only shortest paths, J. Combin. Theory (Ser. B) 41 (1986) 341-355.

[11] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932) 339-362.