

A characterization of halved cubes

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Abstract

The vertex set of a halved cube Q'_d consists of a bipartition vertex set of a cube Q_d and two vertices are adjacent if they have a common neighbour in the cube. Let $d \geq 5$. Then it is proved that Q'_d is the only connected, $\binom{d}{2}$ -regular graph on 2^{d-1} vertices in which every edge lies in two d -cliques and two d -cliques do not intersect in a vertex.

1 Introduction

Let G be a bipartite graph with bipartition $V(G) = X \cup Y$. A *halved graph* G' of G is defined as follows. $V(G') = X$ and $uv \in E(G')$ whenever u and v have a common neighbour in G . G has another halved graph with vertex set Y . When we consider the d -cube Q_d both halved graphs are isomorphic and we talk about the *halved d -cube* Q'_d .

Hamming graphs are exactly those graphs which can be isometrically embedded into a Cartesian product of complete graphs, cf. [2, 8]. In case every one of the factors is the complete graph K_2 on two vertices one obtains

*This work was supported in part by the Ministry of Science and Technology of Slovenia under the grant P1-0206-101.

an isometric embedding into a hypercube and speaks of a binary Hamming graph. If one relaxes the condition of isometry and considers so-called scale embeddings into hypercubes a class larger than that of Hamming graphs arises. It has been characterized by Assouad and Deza [1] as the class of graphs isometrically embeddable into the metric space ℓ_1 . These graphs have in turn be characterized by Deza and Grishukhin [3] and Shpektorov [12] as isometric subgraphs of Cartesian products of complete graphs, cocktail party graphs and halved cubes.

It was this recent study of ℓ_1 -graphs that motivated us to consider halved cubes. As it is clear from the above, halved cubes play an important role in the characterization of ℓ_1 -graphs. In fact, without going into details, by a result of Graham and Winkler [6] about so-called canonical isometric embeddings of graphs into Cartesian products together with an algorithm of Feder [5], a good algorithm for recognizing isometric subgraphs of halved cubes would suffice for a good algorithm for recognizing ℓ_1 -graphs. Unfortunately such an algorithm is not yet known, cf. Deza and Laurent [4]. Aurenhammer, Formann, Idury, Schäffer and Wagner [2] (see also Imrich and Klavžar [8]) proved that it can be decided in $O(mn)$ time whether a given graph on n vertices and m edges is a Hamming graph.

As usual, for a vertex $u \in V(G)$ let $N(u) = \{v; uv \in E(G)\}$. A clique is a maximal complete subgraph. If Q is a clique we will sometimes also use Q to denote its vertex set. A clique on d vertices will be called a *d-clique*. The cocktail party graph on $2n$ vertices is the complete graph K_{2n} minus a complete matching.

In this note we first study the structure of halved cubes and then give a characterization of these graphs. A halved cube on 2^{d-1} vertices is the only connected, $\binom{d}{2}$ -regular graph in which every edge lies in two d -cliques and two d -cliques do not intersect in a vertex.

2 The characterization

We will first summarize several properties of halved cubes. Then we will prove that some of these properties already imply that a given graph is a

halved cube thus obtaining the desired characterization.

The vertex set of the d -cube Q_d may be represented by all sequences of length d over $\{0, 1\}$ where two vertices are adjacent if they differ in exactly one position. We may henceforth consider vertices of the *halved d -cube* Q'_d as sequences of length d over $\{0, 1\}$. In the sequel we will, without loss of generality, assume that a vertex of Q'_d is such a sequence with an even number of 1's. In particular, $(0, 0, \dots, 0) \in V(Q'_d)$. Then two vertices of Q'_d are adjacent iff they differ in two positions.

Clearly, Q'_d has 2^{d-1} vertices. In addition, from the coordinate representation of Q'_d it follows immediately that Q'_d is a $\binom{d}{2}$ -regular graph. (We also recall that halved cubes are distance-regular graphs, cf. [7].)

Note that Q'_3 is isomorphic to the complete graph K_4 on four vertices and that Q'_4 is isomorphic to the cocktail party graph on 8 vertices. To simplify the presentation we may henceforth assume from here on that $d \geq 5$.

Proposition 1 (i) *There are only two types of cliques of Q'_d , namely 4-cliques and d -cliques.*

(ii) *Every vertex of Q'_d lies in d d -cliques.*

(iii) *Q'_d has 2^{d-1} d -cliques.*

Proof. (i) We include the proof of (i) for the sake of completeness although it can be found in [7].

Let u, v and w be distinct vertices of a clique Q of Q'_d . We may, without loss of generality, assume that $u = (0, 0, 0, 0, \dots)$, $v = (1, 1, 0, 0, \dots)$, and $w = (1, 0, 1, 0, \dots)$, where all three vertices agree in the remaining coordinates.

Let z be another vertex of Q . It must have exactly one 1 in its first two coordinates for otherwise it would not be adjacent to at least one of u and v .

If $z = (0, 1, \dots)$, it must agree with w in coordinates 3, 4, \dots , d and there is only one such vertex. Clearly the vertices u, v, w and z induce a clique.

If $z = (1, 0, \dots)$ it must be of the form $(1, 0, 0, \dots, 1, \dots)$. Clearly these $d - 3$ vertices, together with u, v and w form a d -clique.

(ii) By the argument from (i), the d -cliques of Q'_d are induced by the neighbourhoods of vertices of Q_d with an odd number of 1's. Now, since every vertex of Q'_d is in d such neighbourhoods, it is contained in precisely d such cliques.

(iii) This follows by the same argument as (ii). □

We next give properties of halved cubes with respect to a given edge.

Proposition 2 *Let uv be an edge of Q'_d . Then*

(i) $|N(u) \cap N(v)| = 2(d - 2)$.

(ii) uv belongs to precisely two d -cliques of Q'_d , say Q and Q' .

(iii) $Q \cap Q' = \{u, v\}$.

(iv) $Q - \{u, v\}$ and $Q' - \{u, v\}$ are joined by a matching.

Proof. We may assume that $u = (0, 0, 0, 0, \dots, 0)$ and $v = (1, 1, 0, 0, \dots, 0)$. Let w be a vertex adjacent to both u and v . Then w starts out $(1, 0, \dots)$ or $(0, 1, \dots)$ and it has exactly one 1 in the remaining $d - 2$ coordinates. Thus there are $2(d - 2)$ vertices in $N(u) \cap N(v)$. Furthermore, the vertex sets

$$\{u, v, (1, 0, 1, 0, \dots, 0), (1, 0, 0, 1, \dots, 0), \dots, (1, 0, 0, 0, \dots, 1)\}$$

and

$$\{u, v, (0, 1, 1, 0, \dots, 0), (0, 1, 0, 1, \dots, 0), \dots, (0, 1, 0, 0, \dots, 1)\}$$

induce the two cliques containing uv . All the rest now easily follows. □

A connected graph G is a $(0, 2)$ -graph if any two distinct vertices in G have exactly two common neighbors or none at all, cf. [10, 11]. In other words, any two adjacent vertices and any two vertices at distance two must have two common neighbors. Furthermore, if G is bipartite the condition applies only to pairs of vertices at distance two.

We will need the following result due to Mulder [11, page 55], cf. also [9].

Theorem 3 *Let G be a d -regular $(0, 2)$ -graph. Then $|V(G)| = 2^d$ if and only if G is Q_d .*

We are ready now to characterize halved cubes.

Theorem 4 *Let $d \geq 5$. Let G be a connected, $\binom{d}{2}$ -regular graph on 2^{d-1} vertices. Then G is the halved cube Q'_d if and only if*

- (i) *every edge of G is contained in exactly two d -cliques,*
- (ii) *for any d -cliques Q and Q' , $|Q \cap Q'| \neq 1$.*

Proof. If G is a halved cube then Proposition 2 yields (i) and (ii). Conversely, suppose that (i) and (ii) hold. Since G is a $\binom{d}{2}$ -regular graph on 2^{d-1} vertices, $|E(G)| = d(d-1)2^{d-3}$. Thus, because of (i), there are $\frac{2|E(G)|}{\binom{d}{2}} = 2^{d-1}$ d -cliques of G . In addition, since G is $\binom{d}{2}$ -regular and every edge is in two d -cliques, every vertex of G belongs to $\frac{2\binom{d}{2}}{d-1} = d$ d -cliques.

Let Q and Q' be d -cliques of G with $|V(Q) \cap V(Q')| = s$, $s \geq 1$. Then by (ii), $s \geq 2$. Let $u \in Q \cap Q'$ and let $Q, Q', Q_1, Q_2, \dots, Q_{d-2}$ be the d -cliques containing u . Note first that for any i , $Q_i \cap (Q \cap Q') = \{u\}$, for otherwise an edge of this intersection would belong to at least three d -cliques. Thus by (ii), Q_i must intersect $Q \setminus Q'$, $i = 1, 2, \dots, d-2$. Furthermore, if for $w \in Q \setminus Q'$ we have $w \in Q_i \cap Q_j$, $i \neq j$, then the edge uw would not satisfy (i). It follows that $d-s \geq d-2$, thus $s = 2$. Hence if $V(Q) \cap V(Q') \neq \emptyset$ then $|V(Q) \cap V(Q')| = 2$.

Let $n = 2^{d-1}$ and denote the vertices of G by $V(G) = \{u_1, u_2, \dots, u_n\}$. Let H be a graph which we get from G in the following way. To every d -clique Q of G we add a new vertex and join it to every vertex of Q . These are the newly defined edges of H . The original edges of G are all removed. Note that H is bipartite. Since G contains n d -cliques we may write $V(H) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. By construction, $d_H(v_i) = d$, for every $i = 1, 2, \dots, n$, and since every u_i is in d d -cliques, we conclude that H is d -regular.

We claim that H is a $(0, 2)$ -graph. H is connected because G is connected and every edge of G lies in a d -clique. Let $d_H(u_i, u_j) = 2$ and let v_k be a

common neighbor of u_i and u_j . Then $u_i u_j$ must be an edge of G and since it is contained in two d -cliques, there is another common neighbor of u_i and u_j , say v_ℓ . Furthermore, v_k and v_ℓ are their only common neighbors for otherwise $u_i u_j$ would lie in more than two d -cliques of G . Now, let u_k be a common neighbor of vertices v_i and v_j and let Q_i and Q_j be the cliques of G corresponding to v_i and v_j . Since $u_k \in V(Q_i) \cap V(Q_j)$ we have $|V(Q_i) \cap V(Q_j)| = 2$. But this means that v_i and v_j have precisely two common neighbors and the claim is proved.

We have seen that H is a d -regular $(0, 2)$ -graph on 2^d vertices. Thus H is Q_d by Theorem 3. To complete the proof we are going to show that G is the halved graph of H . More precisely, we need to show that $u_i u_j \in E(G)$ if and only if $d_H(u_i, u_j) = 2$. Let $u_i u_j \in E(G)$. Then $u_i u_j$ belongs to a d -clique Q and by construction there is a vertex of H adjacent to every vertex of Q . In particular, $d_H(u_i, u_j) = 2$. Conversely, let $d_H(u_i, u_j) = 2$. Because in H all the edges of G are removed there is a vertex v_k (not in G) such that $u_i v_k \in E(H)$ and $v_k u_j \in E(H)$. But this implies that u_i and u_j belong to a common clique of G , hence $u_i u_j \in E(G)$. \square

We note that condition (ii) of Theorem 4 can be replaced by the following equivalent condition:

(ii') for any d -cliques Q and Q' , $|Q \cap Q'| \leq 2$.

In the proof of Theorem 4 we have shown that (ii) implies (ii'). Suppose now that (ii') holds and assume that $|V(Q) \cap V(Q')| = 1$ for d -cliques Q and Q' . Let $u \in V(Q) \cap V(Q')$. Let $V(Q) = \{u, w_1, w_2, \dots, w_{d-1}\}$. Clearly, $u w_i \in Q$ for $i = 1, 2, \dots, d-1$. Let $Q_i \neq Q$ be the second d -clique containing $u w_i$, $i = 1, 2, \dots, d-1$. Then $Q_i \neq Q'$. Furthermore, if $i \neq j$ then $Q_i \neq Q_j$, for otherwise $|V(Q_i) \cap V(Q_j)| \geq 3$. It follows that u is contained in at least $d + 1$ d -cliques, a contradiction.

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