ON THE STRONG PRODUCT OF A k-EXTENDABLE AND AN l-EXTENDABLE GRAPH

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Abstract

Let $G_1 \otimes G_2$ be the strong product of a k-extendable graph G_1 and an l-extendable graph G_2, and X an arbitrary set of vertices of $G_1 \otimes G_2$ with cardinality $2[(k+1)(l+1)/2]$. We show that $G_1 \otimes G_2 - X$ contains a perfect matching. It implies that $G_1 \otimes G_2$ is $[(k+1)(l+1)/2]$-extendable, whereas the Cartesian product of G_1 and G_2 is only $(k+l+1)$-extendable. As in the case of the Cartesian product, the proof is based on a lemma on the product of a k-extendable graph G and K_2. We prove that $G \otimes K_2$ is $(k+1)$-extendable, and, a bit surprisingly, it even remains $(k+1)$-extendable if we add edges to it.

Introduction

Let us start with the definition of k-extendable graphs. Let G be a graph and k an integer such that $0 \leq k \leq \frac{1}{2}|V(G)| - 1$. Then G is called k-extendable if

(i) G is connected,

(ii) G has a perfect matching (a 1-factor),

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(iii) and any matching in G consisting of k edges can be extended to a perfect matching; in other words, any such matching is a subset of a perfect matching.

Some interesting properties of k-extendable graphs can be found in [3]. We quote just two basic ones we use later. The extendability number of G, $ext(G)$ is the maximum k such that G is k-extendable. A natural problem is to determine the extendability number of different graphs G. For instance, Győri and Plummer [1] studied the Cartesian product of graphs and proved that the Cartesian product of a k-extendable and an l-extendable graph is $(k+l+1)$-extendable and that this bound is sharp for many different types of graphs. In particular, the n-dimensional cube Q_n of 2^n vertices and $n2^{n-1}$ edges, which is the Cartesian product of Q_{k+1} and Q_{l+1} for $n = k + l + 2$, is $(n-1)$-extendable; furthermore, $ext(Q_n) = n - 1$.

In view of these results, it is natural to investigate the extendability numbers of different types of graph products, since such products contain a fair number of 1-factors.

In this paper, we study the extendability number of the strong product of graphs.

Definition. The *strong product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1 is adjacent to v_2, or u_1 is adjacent to u_2 and $v_1 = v_2$, or u_1 is adjacent to u_2 and v_1 is adjacent to v_2. For example, $K_2 \otimes K_2 = K_4$. The "projection" subgraph of $G_1 \otimes G_2$ induced by the vertex set $\{(u, v) : u \in V(G_1), v = v_0 \in V(G_2)\}$ will be denoted by $G_1^{v_0}$. Analogously, $G_1^{V_0}$ denotes the subgraph of $G_1 \otimes G_2$ induced by the vertex set $\{(u, v) : u \in V(G_1), v \in V_0 \subseteq V(G_2)\}$. We similarly use the notation $G_2^{u_0}$ and $G_2^{U_0}$. For a real number x, $[x]_2$ denotes the biggest even integer not greater than x. We will use this notation typically for integers x when $[x]_2 = 2[x/2]$.

Main results

One of the main results of this paper is the following theorem:

Theorem 1. *If G_1 and G_2 are k-extendable and l-extendable graphs ($k, l \geq 0$), respectively, then their strong product $G_1 \otimes G_2$ is $[\frac{(k+1)(l+1)}{2}]$-extendable.*

Remark. One would expect that $(k+1)(l+1)$-extendability (or so) could be proved instead of $[\frac{(k+1)(l+1)}{2}]$-extendability, but this is not the case. Let G_1 and G_2 be two copies of the $(l-1)$-extendable complete bipartite graph $K_{l,l}$. Then $G_1 \otimes G_2$ is at most $2(\sqrt{2} - 1)l^2$-extendable, as we shall see below.

Let $K_{l,l}$ consist of the two sets $X = \{x_1, \ldots, x_l\}$ and $Y = \{y_1, \ldots, y_l\}$, with edges from every element of X to every element of Y. Furthermore, let Z be the subset of X consisting of the elements $\{x_1, \ldots x_m\}$, where $2m \leq l$. Then the neighbors of $Z \times Z$ in $G_1 \otimes G_2$ are the elements of the set $(Z \times Y) \cup (Y \times Z) \cup (Y \times Y)$.

We now consider a partial matching P of the product consisting of the following four types of edges:

$(x_i, y_j)(y_i, y_j)$ for $1 \leq i, j \leq m$, $(x_i, y_j)(y_{i+m}, y_j)$ for $1 \leq i \leq m, m < j \leq l$,
$(y_i, x_j)(y_i, y_{j+m})$ for $1 \leq i, j \leq m$, $(y_i, x_j)(y_i, y_j)$ if $m < i \leq l, 1 \leq j \leq m$.

Clearly this matching uses all neighbors of $Z \times Z$ in $(Z \times Y) \cup (Y \times Z)$. Thus, if we wish to extend this matching to a complete matching, we have to match all vertices of $Z \times Z$ with vertices in $Y \times Y$ not already used by those of the matching P. The number of these vertices is $l^2 - 2ml$. So P can definitely not be extended if $l^2 - 2ml < m^2$. Thus $G_1 \otimes G_2$ is not $2\lceil(\sqrt{2}-1)l\rceil l$-extendable.

Note that the numerical value for this bound is $0.8284\, l^2$ and that the smallest nonextendable P found by this construction for $l = 11$ has $m = 5$. The size of P in this case is $0.8333\, l^2$.

In view of this example, we do not have a good conjecture for a sharp result either. We will strengthen Theorem 1 in a different sense.

In the special case $G_2 = K_2$ a higher extendability number can be proved.

Lemma 1. *Let G be a k-extendable graph. Then $G \otimes K_2$ is $(k+1)$-extendable.*

Lemma 1 is sharp in the sense that there are infinitely many k-extendable graphs for each k such that $G \otimes K_2$ is not $(k+2)$-extendable. Take an arbitrary k-extendable graph G with minimum degree $k+1$. If $d_G(u_0) = k+1$ and $N(u_0) = \{u_1, ..., u_{k+1}\}$, then at least one neighbor of u_0, say u_1, has a neighbor $u_{k+2} \neq u_0, u_2, ..., u_{k+1}$, since G is not a complete graph on $k+2$ vertices. The edges $(u_0, 2)(u_1, 2), (u_1, 1)(u_{k+2}, 1), (u_2, 1)(u_2, 2)$, $(u_3, 1)(u_3, 2), ..., (u_{k+1}, 1)(u_{k+1}, 2)$ do not extend to a perfect matching of $G \otimes K_2$ since, if we delete the $2k+4$ endvertices of these edges, then the vertex $(u_0, 1)$ will be an isolated vertex in the remaining graph.

On the other hand, we prove an even stronger statement than Lemma 1. This stronger statement will play a key role in the proof of the main theorem. The strengthening of Lemma 1 is as follows:

Lemma 2. *Let G be a k-extendable graph. Then $G \otimes K_2$ is $(k+1)$-extendable. Furthermore, all graphs obtained from $G \otimes K_2$ by adding an arbitrary number of edges are*

$(k+1)$-*extendable as well.*

Remark 1. The statement of Lemma 2 is pretty strong. Note that adding an edge to a k-extendable bipartite graph may make it non-bipartite, and the resulting graph need not even be 1-extendable any more. Also the addition of an edge to a k-extendable non-bipartite graph will result in a $(k-1)$-extendable graph that is not necessarily k-extendable (see [5]).

Remark 2. The property described in Lemma 2 says that the graph that remains after the deletion of any $2k+2$ vertices from $G \otimes K_2$ has a perfect matching. This property can be considered as an extension of factor-criticality. So we say that $G \otimes K_2$ is $(2k+2)$-*factor-critical.*

Using Lemma 2, we can prove a result that is even stronger than Theorem 1:

Theorem 2. *If G_1 and G_2 are k-extendable and l-extendable graphs ($k, l \geq 0$), respectively, then their strong product $G_1 \otimes G_2$ is $[(k+1)(l+1)]_2$-factor-critical, that is, for any set X of $[(k+1)(l+1)]_2$ vertices the graph $G_1 \otimes G_2 - X$ has a perfect matching.*

Remark 3. Theorem 2 is almost sharp, since the statement does not hold for $[(k+2)(l+2)]_2$ vertices. To see this, take an arbitrary k-extendable graph G_1 containing a vertex x of degree $k+1$ (there are infinitely many graphs like that, for instance $K_{k+1,k+1}$ in which all degrees are $k+1$) and an arbitrary l-extendable graph with a vertex y of degree $l+1$. Then the degree of (x, y) in the strong product is $(k+2)(l+2)-1 \leq [(k+2)(l+2)]_2$, and if X contains all the neigbours of (x, y), then $G_1 \otimes G_2 - X$ obviously does not have a perfect matching.

We make the following conjecture:

Conjecture. *If G_1 and G_2 are k-extendable and l-extendable graphs ($k, l \geq 0$), respectively, then their strong product $G_1 \otimes G_2$ is $([(k+2)(l+2)]_2 - 2)$-factor-critical, that is, for any set X of $[(k+2)(l+2)]_2 - 2$ vertices the graph $G_1 \otimes G_2 - X$ has a perfect matching.*

Proofs

We begin with the proof of Lemma 2, which implies Lemma 1.

Proof of Lemma 2.

We prove the statement by contradiction. If the statement does not hold then, by Tutte's [4] famous theorem about 1-factors, there is an $l \geq 0$ and there are $2k+2+l$

vertices $v_1, ..., v_{2k+2+l}$ such that their deletion results in a graph that has at least $l+1$ odd components.

Notice that for any vertex $v \in V(G)$, the vertices $(v,1)$ and $(v,2)$ have the same neighbors apart from each other and so they belong to the same component, unless we deleted at least one of them. Thus each of the odd components contains a vertex (v,i), $(i = 1$ or $2)$ such that $(v, 3-i)$ is deleted from $G \otimes K_2$. It follows that we deleted at most k full vertex pairs $\{(v,1),(v,2)\}$. The graph G is k-extendable and thus $(k+1)$-connected (see [3]). For $v \in V(G)$, $(v,1)$ and $(v,2)$ have the same neighbors in $G \otimes K_2$ (apart from $(v,1)$ and $(v,2)$). Since we deleted at most k vertex pairs $(v,1)$, $(v,2)$ completely, the remaining graph is connected, a contradiction. []

Proof of Theorem 2.

We prove the statement by induction on $k+l$.

First, suppose that $k = 0$. Notice that the statement trivially holds if $l = 0$ as well. So we assume that $l > 0$. We prove this case by induction on t, where $2t = |V(G_1)|$. The case $t = 1$ follows from Lemma 2. Let us fix a perfect matching $\{u_1u_2, u_3u_4, ..., u_{2t-1}u_{2t}\}$ in G_1. Extend it to a spanning tree of G_1 and contract the edges $u_1u_2, u_3u_4, ..., u_{2t-1}u_{2t}$ of the matching. The spanning tree is transformed into a spanning tree of the contracted graph and contains a vertex of degree one. Suppose that, say, the vertex obtained from the contraction of u_1u_2 is a vertex of degree one. It implies that $G_1 - \{u_1, u_2\}$ is connected and has a perfect matching $\{u_3u_4, ..., u_{2t-1}u_{2t}\}$, in other words, it is 0-extendable. G_1 is connected, so we may assume that, say, u_1 has a neighbor in $\{u_3, u_4, ..., u_{2t-1}, u_{2t}\}$. Let $X = \{x_1, ..., x_{[l+1]_2}\}$ be an arbitrary vertex set in $G_1 \otimes G_2$ and $X_0 = X \cap V(G_2^{u_1,u_2})$. Set $X_1 = X_0$ if $|X_0|$ is even and $X_1 = X_0 \cup \{(u_1, v)\}$ if $|X_0|$ is odd, where v is an arbitrary vertex in G_2 such that $(u_1, v) \notin X$. (There is such a v since $|V(G_2)| \geq 2l+2$ but $|X| \leq l+1$.) By Lemma 2 the graph $G_2^{u_1,u_2} - X_1$ contains a perfect matching M, since $|X_1| \leq l+1$. Note that if v has to be taken, then the vertex (u_1, v) has at least $l+2$ neighbors in $G_1 \otimes G_2 - V(G_2^{u_1,u_2})$ by the choice of u_1, the definition of the strong product, and since an l-extendable graph is $(l+1)$-connected (see [3]). Thus there exists a vertex $(u, w) \in V(G_1 \otimes G_2) - V(G_2^{u_1,u_2})$ such that $(u, w) \notin X$ and $(u_1, v)(u, w) \in E(G_1 \otimes G_2)$. Then, let $X' = (X - V(G_2^{u_1,u_2})) \cup \{(u, w)\}$. Now $|X'|$ is even and at most $[l+1]_2$. The graph $G_1 - \{u_1, u_2\}$ is still 0-extendable. So, by the induction hypothesis for $G_1 \otimes G_2 - V(G_2^{u_1,u_2}) = (G_1 - \{u_1, u_2\}) \otimes G_2$ and X', there exists a perfect matching M_1 in $G_1 \otimes G_2 - V(G_2^{u_1,u_2}) - X'$. Then $M_1 \cup M \cup \{(u_1, v)(u, w)\}$ is a perfect matching in $G_1 \otimes G_2 - X$.

From now on, we may assume that $k, l \geq 1$, because of the symmetry of k and l.

Now let us consider the strong product $G_1 \otimes G_2$ of a k-extendable graph G_1 and an l-extendable graph G_2, where $k + l > 1$. Let $X = \{x_1, ..., x_{[(k+1)(l+1)]_2}\}$ be an arbitrary set of vertices in $G_1 \otimes G_2$. We distinguish two cases with respect to the size of the vertex sets $X \cap V(G_2^u)$.

Case 1. There is an edge u_1u_2 in $E(G_1)$ for which $|X \cap V(G_2^{u_1,u_2})| \geq [l+2]_2$.

Take $[l+2]_2$ vertices, say $X_0 = \{x_1, ..., x_{[l+2]_2}\}$ in $X \cap V(G_2^{u_1,u_2})$. Then $G_2^{u_1,u_2} - X_0$ contains a perfect matching M by Lemma 2. Consider the edges $y_1z_1, ..., y_pz_p$ of M such that $z_i \in X - X_0$ and $y_i \notin X - X_0$. Note that the degree of every vertex in $G_1 \otimes G_2$ is at least $(k+2)(l+2) - 1$ by the $(k+1)$-connectivity of G_1, the $(l+1)$-connectivity of G_2, and by the definition of the strong product. It also implies that every vertex y_i has at least $k(l+2)$ neighbors in $G_1 \otimes G_2 - V(G_2^{u_1,u_2})$. Since $G_2^{u_1,u_2}$ contains at least $[l+2]_2 + p$ elements of X, we infer that $G_1 \otimes G_2 - V(G_2^{u_1,u_2}) - X$ contains at least $k(l+2) - ([(k+1)(l+1)]_2 - [l+2]_2 - p) \geq p$ neighbors of any y_i. Thus there exist vertices $w_1, ..., w_p \in V(G_1 \otimes G_2) - V(G_2^{u_1,u_2})$ such that $w_i \notin X$, $y_iw_i \in E(G_1 \otimes G_2)$ for $i = 1, ..., p$. Then, let $X_1 = (X - V(G_2^{u_1,u_2})) \cup \{w_1, ..., w_p\}$. Now $|X_1|$ is even and at most $[(k+1)(l+1)]_2 - [l+2]_2 \leq [k(l+1)]_2$. The graph $G_1 - \{u_1, u_2\}$ is $(k-1)$-extendable (see [3]). So, by the induction hypothesis for $G_1 \otimes G_2 - V(G_2^{u_1,u_2}) = (G_1 - \{u_1, u_2\}) \otimes G_2$ and X_1, there exists a perfect matching M_1 in $G_1 \otimes G_2 - V(G_2^{u_1,u_2}) - X_1$. If M_0 denotes the set of edges of M with both endvertices in X then $M_1 \cup (M - M_0) \cup \{y_1w_1, ..., y_pw_p\} - \{y_1z_1, ...y_pz_p\}$ is a perfect matching in $G_1 \otimes G_2 - X$.

Case 2. For every pair of vertices u_i and u_j in G_1 such that $u_iu_j \in E(G_1)$, we have $|X \cap V(G_2^{u_i,u_j})| \leq [l]_2 + 1$.

Let us fix a perfect matching $\{u_1u_2, u_3u_4, ..., u_{2t-1}u_{2t}\}$ in G_1 where $2t = |V(G_1)|$.

Consider the case $l = 1$ first. By assumption $|X \cap V(G_2^{u_i,u_j})| \leq 1$ for every u_i and u_j such that $u_iu_j \in E(G_1)$, and $|X| = 2k + 2$. Let I denote the set of indices i such that $|X \cap V(G_2^{u_{2i-1},u_{2i}})| = 1$. Without loss of generality, we may assume that $X \cap V(G_2^{u_{2i-1}})$ is empty for every i. Clearly, $|I| = 2k + 2$. Let us divide I into two disjoint parts I_1 and I_2 of size $k + 1$. The graph G_1 is k-extendable and therefore $(k+1)$-connected (see [3]). Thus, by Menger's theorem, there exists a collection of $k+1$ pairwise vertex disjoint paths from $U_1 = \{u_{2i} : i \in I_1\}$ to $U_2 = \{u_{2i} : i \in I_2\}$. Let F denote the forest consisting of these paths.

We will construct a matching M of $|E(F)|$ edges in $G_1 \otimes G_2 - X$ by taking one (and only one) edge joining $V(G_2^{u_i})$ and $V(G_2^{u_j})$ if $u_i u_j \in E(F)$. We select the edges of M one by one. Suppose that $u_i u_j \in E(F) \subseteq E(G_1)$ is the next edge and we are to choose the appropriate edge in $V(G_2^{u_i,u_j})$. The vertex set $X \cap V(G_2^{u_i,u_j})$, together with the so far chosen edges of M, covers at most one vertex in each of $G_2^{u_i}$ and $G_2^{u_j}$ by the choice of F. So there is a $v \in V(G_2)$ such that $(u_i, v), (u_j, v)$ are not covered yet. Put this edge $(u_i, v)(u_j, v)$ into M. Proceed similarly for the other edges of F. Then $|(X \cup V(M)) \cap V(G_2^{u_{2i-1},u_{2i}})|$ is even and at most 4 ($i = 1, 2, ..., t$), and so $G_2^{u_{2i-1},u_{2i}} - X - V(M)$ has a perfect matching by Lemma 2 ($i = 1, 2, ..., t$). Then the union of these t matchings and M is a desired perfect matching in $G_1 \otimes G_2 - X$.

From now on, we may assume that $k, l \geq 2$ because of the symmetry of k and l. Furthermore, we assume that $|V(G_2)| = 2s \geq 2l + 4$. (We settle the case $|V(G_2)| = 2l + 2$ separately.) We will use the following claim which makes it possible to apply an idea similar to the one used in the proof of the case $l = 1$.

Claim. *There exists a spanning forest F of G_1 such that, if $d_F(x)$ denotes the degree of x in F, then:*

(1) $d_F(u_{2i-1}) + d_F(u_{2i}) + |X \cap V(G_2^{u_{2i-1},u_{2i}})| \leq 2l + 2$ *for* $i = 1, ..., t$,

(2) $d_F(u_{2i-1}) + d_F(u_{2i}) + |X \cap V(G_2^{u_{2i-1},u_{2i}})|$ *is even for* $i = 1, 2, ..., t$,

(3) $d_F(u_i) + d_F(u_j) + |X \cap V(G_2^{u_i,u_j})| \leq 2s + l + 2$ *for* $u_i u_j \in E(G_1)$ *where* $2s = |V(G_2)|$,

(4) the union of F with the edges $u_1u_2, ..., u_{2t-1}u_{2t}$ is acyclic.

Proof of the Claim. Let $d_X(u)$ denote $|X \cap V(G_2^u)|$ for $u \in V(G_1)$. Notice that the empty forest obviously satisfies (1), (3), and (4). Starting with the empty forest, we change this forest subgraph F of G_1 step by step so that the number of indices i for which $d_F(u_{2i-1}) + d_F(u_{2i}) + d_X(u_{2i-1}) + d_X(u_{2i})$ is odd decreases by two in each step. Suppose that some forest F has been chosen already. Suppose that there are (an even number of) vertex pairs $\{u_{2i-1}, u_{2i}\}$ such that $d_F(u_{2i-1}) + d_F(u_{2i}) + d_X(u_{2i-1}) + d_X(u_{2i})$ is odd. Take two such vertex pairs $\{u_{2i-1}, u_{2i}\}$ and $\{u_{2j-1}, u_{2j}\}$.

We prove that there exists a path P from $\{u_{2i-1}, u_{2i}\}$ to $\{u_{2j-1}, u_{2j}\}$ such that the symmetric difference of the edge sets $E(F)$ and $E(P)$, maybe after deletion of some edges, is a new forest still satisfying (1), (3), (4), and (2) for two more indices.

Obviously, the path cannot use a vertex u if it belongs to a vertex pair $\{u_{2m-1}, u_{2m}\}$ such that $d_F(u_{2m-1}) + d_F(u_{2m}) + |X \cap V(G_2^{u_{2m-1},u_{2m}})| \geq 2l + 1$ unless we have to join this pair to another pair when this sum is odd, and so precisely $2l + 1$, and the sum will increase by one only. Similarly, the path cannot use both members of a vertex pair $\{u_i, u_j\}$ such that $d_F(u_i) + d_F(u_j) + |X \cap V(G_2^{u_i,u_j})| \geq 2s + l - 1$ and $u_iu_j \in E(G_1)$.

However, as we shall see, it is harmless if the path P uses both members of a vertex pair $\{u_{2i-1}, u_{2i}\}$ with $2l \geq d_F(u_{2i-1}) + d_F(u_{2i}) + |X \cap V(G_2^{u_{2i-1},u_{2i}})| \geq 2l - 1$.

Now, let a be the number of vertex pairs $\{u_{2i-1}, u_{2i}\}$ such that $d_F(u_{2i-1}) + d_F(u_{2i}) + |X \cap V(G_2^{u_{2i-1},u_{2i}})| \geq 2l + 1$ and let A denote the set of these $2a$ vertices. Let B denote the set of vertices u in $V(G_1) - A$ such that $d_F(u) + d_X(u) \geq l + 3$ and let $b = |B|$.

First suppose that $a + b \leq k$. Then $G_1 - A$ is $(k - a)$-extendable (see [3]) and so it is $(k - a + 1)$-connected (see [3] again). Therefore $G_1 - A - B$ is still connected. Then there is a path P from $\{u_{2i-1}, u_{2i}\}$ to $\{u_{2j-1}, u_{2j}\}$ avoiding $A \cup B$. Suppose the path P uses both members of some d vertex pairs $\{u_{2m-1}, u_{2m}\}$ with $d_F(u_{2m-1}) + d_F(u_{2m}) + |X \cap V(G_2^{u_{2m-1},u_{2m}})| \geq 2l - 1$. Then these $2d$ vertices divide the path P into $2d + 1$ segments. Delete the edge set of the 2-nd, 4-th,...,$2d$-th segment of P. The obtained smaller edge set E_P still satisfies the conditions it satisfied before and the sum of the F-degrees of the vertices u_{2m-1}, u_{2m} increases by two only.

Consider the symmetric difference F_0 of the edge sets E_P and $E(F)$. If F_0 contains a cycle, then delete the edges of this cycle from F_0 and continue until it becomes acyclic. It is easy to see that the resulting edge set defines a desired forest F. Notice that we may assume that F_0 does not contain any edge $u_{2m-1}u_{2m}$, since we can delete these edges without violating any requirement. Even more, we may assume that F_0 is acyclic and remains acyclic even if we add the edges $u_{2m-1}u_{2m}$, where $m = 1, ..., t$. For, suppose that addition of all the edges $u_{2m-1}u_{2m}$ produces cycles. As long as we have a cycle, delete its edges. Then delete the remaining edges $u_{2m-1}u_{2m}$ from the remaining forest. Obviously, we obtain another (smaller) desired F_0. Thus there is an F_0 still satisfying (1), (4), and (2) for two more indices. We have to check only (3).

Let $u_i, u_j \in V(G_1)$ be vertices such that $u_iu_j \in E(G_1)$. Naturally, if $\{u_i, u_j\} \subseteq (A \cup B)$, then nothing changed, (3) still holds. If $\{u_i, u_j\} \cap (A \cup B) = \emptyset$, then $d_F(u_i) + d_F(u_j) + |X \cap V(G_2^{u_i,u_j})| \leq 2l + 4 \leq 2s + l - 2$ by the choice of B and since $2s \geq 2l + 4$ and $l \geq 2$. So $d_{F_0}(u_i) + d_{F_0}(u_j) + |X \cap V(G_2^{u_i,u_j})| \leq 2s + l + 2$.

Suppose that $\{u_i, u_j\} \cap (A \cup B) = \{u_j\}$. We have $d_{F_0}(u_j) + d_X(u_j) = d_F(u_j) + d_X(u_j) \leq 2l + 2$ because of (1) for the vertex pair containing u_j. Moreover $d_F(u_i) + d_X(u_i) \leq l + 2$ by the choice of B, and so $d_{F_0}(u_i) + d_X(u_i) \leq l + 4$. Thus $d_{F_0}(u_j) + d_X(u_j) + d_{F_0}(u_i) + d_X(u_i) \leq 3l + 6 \leq 2s + l + 2$.

Now suppose that $a + b \geq k + 1$. In this case we derive a contradiction. Contracting each edge $u_{2i-1}u_{2i}$ of G_1 to a vertex w_i, G_1 is transformed into a graph H of t vertices and F is transformed into a forest F' by (4). Let A' and B' denote the sets of vertices w_i such that $A \cap \{u_{2i-1}, u_{2i}\}$ and $B \cap \{u_{2i-1}, u_{2i}\}$ are nonempty, respectively. Let z denote the number of indices i such that $|X \cap V(G_2^{u_{2i-1},u_{2i}})|$ is odd. Then the number of vertices w in H such that $d_{F'}(w) = 1$ is at most $z - 2$ because of the construction of F. (We joined these odd pairs preserving evenness if it was reached earlier and we still have to join two or more pairs.)

Naturally, $z \leq [(k+1)(l+1)]_2$, and the number of vertices w in $H - A' - B'$ such that $d_{F'}(w) = 1$ is at most $[(k+1)(l+1)]_2 - 2 - \sum_{w_i \in A' \cup B'} |X \cap V(G_2^{u_{2i-1},u_{2i}})|$. However, $d_{F'}(w) \geq l \geq 2$ for $w \in A'$ and $d_{F'}(w) \geq 2$ for $w \in B'$ because of the definition of A and B and since we are in Case 2. So F' has at most $[(k+1)(l+1)]_2 - 2 - \sum_{w_i \in A' \cup B'} |X \cap V(G_2^{u_{2i-1},u_{2i}})|$ vertices of degree 1. It is an easy exercise to show that in any forest with at least one edge, $\sum_{d(x) \geq 2}(d(x) - 2)$ is less by at least 2 than the number of vertices of degree 1. However,

$$\sum_{w \in A' \cup B'} (d_{F'}(w) - 2) \geq a(2l - 1) + b(l + 1) - \sum_{w_i \in A' \cup B'} |X \cap V(G_2^{u_{2i-1},u_{2i}})|$$

$$\geq a(l + 1) + b(l + 1) - \sum_{w_i \in A' \cup B'} |X \cap V(G_2^{u_{2i-1},u_{2i}})|$$

$$\geq (k + 1)(l + 1) - \sum_{w_i \in A' \cup B'} |X \cap V(G_2^{u_{2i-1},u_{2i}})|,$$

a contradiction. This completes the proof of the claim.

Finally, consider the case $|V(G_2)| = 2l + 2$. By symmetry, we may assume that $|V(G_1)| = 2k + 2$ also holds, otherwise the above proof applies for k rather than l. Recall that we are still in Case 2, thus we have $|X \cap V(G_2^{u_i,u_j})| \leq [l]_2 + 1$ for every u_i and u_j such that $u_iu_j \in E(G_1)$ and we fixed a perfect matching $\{u_1u_2, u_3u_4, ..., u_{2k+1}u_{2k+2}\}$ in G_1, since $|V(G_1)| = 2k + 2$. Adding the above inequalities for the edges $u_1u_2, u_3u_4, ...,$ $u_{2k+1}u_{2k+2}$, we obtain $|X| = [(k+1)(l+1)]_2 \leq (k+1)([l]_2 + 1)$. It follows that l is even. A similar argument shows that k is even as well. Furthermore, $|X \cap V(G_2^{u_{2i-1},u_{2i}})| = l + 1$ for all the edges $u_1u_2, u_3u_4, ..., u_{2k+1}u_{2k+2}$ except one, say u_1u_2. G_1 is k-extendable, so it is $(k + 1)$-connected and $k \geq 2$. Then $G_1 - \{u_1, u_2\}$ is connected, an edge leads from $\{u_3, u_4\}$ to the rest, say $u_3u_5 \in E(G_1)$. We have $|X \cap V(G_2^{u_3,u_5})| \leq l + 1$ which implies the existence of a vertex $v \in V(G_2)$ such that $(u_3, v), (u_5, v) \notin X$. Then $G_2^{u_3,u_4} - X - \{(u_3, v)\}$

contains a perfect matching M_1 by Lemma 2, and similarly, $G_2^{u_5,u_6} - X - \{(u_5, v)\}$ contains a perfect matching M_2. Furthermore, $G_1 - \{u_3, u_4, u_5, u_6\}$ is $(k-2)$-extendable and $X_0 = X - V(G_1^{u_3,u_4,u_5,u_6})$ has cardinality $[(k-1)(l+1)]_2$. So we can apply the induction hypothesis for $G_1 \otimes G_2 - V(G_1^{u_3,u_4,u_5,u_6}) = (G_1 - \{u_3, u_4, u_5, u_6\}) \otimes G_2$ and X_0 to see that the graph $(G_1 - \{u_3, u_4, u_5, u_6\}) \otimes G_2 - X_0$ contains a perfect matching M_3. Then $M_1 \cup M_2 \cup M_3 \cup \{(u_3, v)(u_5, v)\}$ is a desired perfect matching in $G_1 \otimes G_2 - X$.

Now we are ready to settle Case 2. Let us take a matching M of $|E(F)|$ edges in $G_1 \otimes G_2 - X$ such that we take one (and only one) edge joining $V(G_2^{u_i})$ and $V(G_2^{u_j})$ if $u_i u_j \in E(F)$. We take the edges of M one by one. Suppose that $u_i u_j \in E(F) \subseteq E(G_1)$ is the next edge and that we are to choose the appropriate edge in $V(G_2^{u_i,u_j})$. The vertex set $X \cap V(G_2^{u_i,u_j})$ together with the so far chosen edges of M cover a set Y of at most $2l+2$ vertices in $G_2^{u_i,u_j}$ by (1). If there is a $v \in V(G_2)$ such that $(u_i, v), (u_j, v) \notin Y$, then take the edge $(u_i, v)(u_j, v)$. Suppose that there is no such vertex $v \in V(G_2)$. Since G_2 is l-extendable and thus $|V(G_2)| = 2s \geq 2l+2$, it follows that the set Y_0 of vertices v such that both (u_i, v) and (u_j, v) are in Y has cardinality at most l. Conditions (1) and $|V(G_2)| = 2s \geq 2l+2$ imply that $V(G_2^{u_i}) \not\subseteq Y$ and $V(G_2^{u_j}) \not\subseteq Y$. Let Y_i and Y_j denote the nonempty set of vertices $v \in V(G_2)$ such that $(u_i, v) \notin Y$ and $(u_j, v) \notin Y$, respectively. Since G_2 is $(l+1)$-connected and since Y_0 has at most l elements, there is an edge $v_1 v_2 \in V(G_2)$ such that $v_1 \in Y_i, v_2 \in Y_j$. Then take the edge $(u_i, v_1)(u_j, v_2)$. Proceed similarly for the other edges of F. Then $|(X \cup V(M)) \cap V(G_2^{u_{2i-1},u_{2i}})|$ is even and at most $2l+2$ $(i = 1, 2, ..., t)$ by the Claim and so $G_2^{u_{2i-1},u_{2i}} - X - V(M)$ has a perfect matching by Lemma 2 $(i = 1, 2, ..., t)$. Then the union of these t matchings and M is a desired perfect matching in $G_1 \otimes G_2 - X$. This completes the proof of Theorem 2. []

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