A theorem on integer flows on Cartesian products of graphs

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ABSTRACT

It is shown that the Cartesian product of two nontrivial graphs admits a nowhere-zero 4-flow. If both factors are bipartite, then the product admits a nowhere-zero 3-flow. © *John Wiley & Sons, Inc.*

Keywords: Integer flows, Cartesian products

1. INTRODUCTION

All graphs considered in this note are simple and connected. A graph is *trivial* if it is isomorphic to K_1. Denote the path and the cycle of length n by P_n and C_n, respectively. A graph is *even* if all of its vertices are of even degree. Such graphs are also called Eulerian.

Let D be an orientation of a graph G, and let $f : E(G) \to \mathbb{Z}$, such that $-k < f(e) < k$ for every $e \in E(G)$. The pair (D, f) is a *k-flow* of G if Kirchhoff's condition

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e),$$

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[*] Supported in part by the Ministry of Science and Technology of Slovenia, Research Project P1–0210–101–00.

is satisfied at every vertex $v \in V(G)$. (Here $E^+(v)$ and $E^-(v)$ denote the sets of outgoing and incoming edges – with respect to the orientation D – incident with v, respectively.) A k-flow (D, f) is *nowhere-zero* (or shortly *NZ*) if $f(e) \neq 0$ for every $e \in E(G)$.

The concept of nowhere-zero k-flows was introduced by Tutte [1, 2]. Tutte [1] proved that a cubic graph admits a nowhere- zero 3-flow if and only if it is bipartite. Tutte [2] also proved that a plane graph admits a nowhere-zero k-flow if and only if it is face k-colorable. Moreover, every Eulerian graph clearly admits a nowhere-zero 2-flow. For further results on nowhere-zero flows we recommend the book [3] by Zhang.

We will consider nowhere-zero flows on Cartesian products. To fix notation, let us recall that the Cartesian product $G \,\square\, H$ of two graphs G and H is defined on the Cartesian product $V(G) \,\square\, V(H)$ of the sets of vertices of the factors and that two vertices (x_1, x_2), (y_1, y_2) of $G \,\square\, H$ are adjacent if $x_1 = y_1$ and $[x_2, y_2] \in E(H)$, or if $[x_1, y_1] \in E(G)$ and $x_2 = y_2$. Edges of the type $[(x_1, x_2), (x_1, y_2)]$ are called *H-edges*, the others *G-edges.* We also say that $[(x_1, x_2), (x_1, y_2)]$ *corresponds* to the edge $[x_2, y_2]$ of H.

Let $v = (v_1, v_2) \in G \,\square\, H$. Then the subgraph induced in $G \,\square\, H$ by the vertices of the form

$$\{(v_1, h) | h \in V(H)\}$$

is the *H-layer* of $G \,\square\, H$ through v and denoted by H^v or H^{v_1}. Analogously we define the G-layer through v.

This paper is motivated by the following observation, which is most likely known.

Proposition 1. The Cartesian product $G \,\square\, H$ of two graphs G and H admits a NZ k-flow, if both factors admit a NZ k-flow. Moreover every hypercube Q_n $(n \geq 2)$ admits a NZ 3-flow.

Proof. Let (D_1, f_1) and (D_2, f_2) be NZ k-flows of G and H, respectively. Define a flow (D, f) of $G \,\square\, H$ as follows: If an edge e of $G \,\square\, H$ corresponds to an edge of e_1 of G, then set $D(e) = D_1(e_1)$ and $f(e) = f_1(e_1)$. Similarly, if e corresponds to an edge e_2 of H, then set $D(e) = D_2(e_2)$ and $f(e) = f_2(e_2)$. Clearly (D, f) is a NZ k-flow of G.

For the second assertion note first that Q_n is Eulerian if n is even, and so admits a NZ 2-flow (and thus a NZ 3-flow). Furthermore, Q_3 is a bipartite cubic graph, and thus admits a NZ 3-flow by the theorem of Tutte [1] mentioned above. Finally, the observation that $Q_{2k+1} = Q_{2k-2} \,\square\, Q_3$ and an application of the first assertion completes the proof for odd $n \geq 5$. ∎

2. THE MAIN RESULT

For the proof of the main result we need Cartesian products from which certain edges have been removed. Such products will be called generalized Cartesian products and are defined below. For the definition we partition the vertex sets of the factors into two subsets each, one consisting of *active*, the other of *inactive* vertices.

The generalized Cartesian product $G_1 \square_w G_2$ of two graphs has the same vertex set as $G_1 \square G_2$ and its edge-set arises from that of $G_1 \square G_2$ by removal of all edges in layers $G_i^{v_{3-i}}$, $i \in \{1, 2\}$, for which v_i is an inactive vertex of G_i.

Theorem 1. Let $G \square H$ be the Cartesian product of two nontrivial graphs. Then $G \square H$ admits

(i) a NZ 3-flow if both factors are bipartite, and
(ii) a NZ 4-flow in general.

Proof. Suppose first that G is a nontrivial graph, all of whose vertices are active. We define an operation $(*)$ and apply it repeatedly to G until the active vertices of the resulting graph G^* are of degree < 3.

$(*)$ Let v be an active vertex of degree ≥ 3. Then split it into two vertices v_1 and v_2 of degree 2 and $d(v) - 2$, respectively. Moreover let v_1 be an inactive and v_2 an active vertex of the resulting graph.

Since all inactive vertices introduced by $(*)$ are of degree 2, we infer that the components of G^* are cycles and paths. Clearly every such path has active end vertices. We would like G^* to have the following property:

(I) The number of active vertices of every cycle of G^* is different from 1.

Clearly G has property (I). In what follows, we will describe how one can preserves property (I) by applications of (*).

Let *n-loop* be a graph consisting of n cycles with precisely one common vertex, and suppose that this is the only active vertex of that graph. Note that if in the procedure of constructing G^*, we can avoid the generation of n-loops, then the property (I) is preserved.

Suppose that $\widehat{G}$ is a graph with no n-loops constructed by repeated application of (*) and that $v \in V(\widehat{G})$ is an active vertex of degree ≥ 3. Suppose also that after applying $(*)$ at v, we obtain a graph $\widehat{G}$ that contains an m- loop (for some m). Clearly, v_1 or v_2 is a vertex of this m- loop. Let the vertex v_1 be adjacent to vertices x_1, x_2, and y be a vertex adjacent to v_2 in $\widehat{G}$. If both v_1 and v_2 are in this component, then it is easy to see that v belongs to an $(m+1)$-loop of $\widehat{G}$. So we may assume that v_1 and v_2 belong to different components of $\widehat{G}$.

Suppose now that v_1 belongs to an m-loop. Then the component of v_1 contains an active vertex different from v_2. Denote by G' the graph obtained from $\widehat{G}$ by replacement of the edges v_1x_1 and v_2y by v_1y and v_2x_1, respectively. Clearly G' can be constructed from $\widehat{G}$ using (*) at v. Note that v_1 and v_2 belong to the same component of G' and that this component contains at least two active vertices. So G' does not contain an n-loop.

Finally suppose that the component that contains v_1 is not an m- loop but the component of v_2 is. Then the component containing v_1 cannot be a cycle with no active vertex, otherwise v would have to belong to an $(m+1)$-loop in $\widehat{G}$. Now, construct G' as described above and observe that it must be n-loop free.

For the remainder of the proof we shall make use of the following, obvious claim:

(II) Let H be a graph with active and inactive vertices. If the graph $G^* \,\square_w\, H$ admits a NZ k-flow, then also $G \,\square_w\, H$.

Now, let G_1 and G_2 be two nontrivial graphs and let G_1^* and G_2^* be graphs constructed as above from G_1 and G_2, respectively. Consider the product $H_1 \,\square_w\, H_2$ where H_i is a component of G_i for $i \in \{1,2\}$. Since components of G_1^* and G_2^* are paths with active end vertices and cycles with property (I), it follows that $H_1 \,\square_w\, H_2$ is homeomorphic to one of graphs $P_n \,\square\, P_m$ or $P_n \,\square\, C_m$ or $C_n \,\square\, C_m$ with nontrivial factors.

Note that every $C_n \,\square\, C_m$ is an even graph, and so admits a NZ 2- flow. Consider now the complete grid $P_n \,\square\, P_m$ $(m, n \geq 2)$ as a plane graph of which each face, except maybe the outer one, is a 4- cycle. It is easy to see that the faces of this graph can be properly colored using three colors. Now we invoke the theorem of Tutte cited in the introduction to see that this graph admits a NZ 3-flow.

Similarly, consider $C_n \,\square\, P_m$ $(m \geq 1)$ as a plane graph such that all faces except at most two are 4-cycles. Faces of this graph can be colored with three colors if n is even, otherwise they can be colored with four colors. Therefore this graph admits a NZ 3-flow for n even and it admits a NZ 4-flow for n odd.

Now, by (II), it is easy to see that $G_1 \,\square\, G_2$ admits a NZ 4-flow. Moreover if G_1 and G_2 are bipartite, then each cycle of G_1^* and G_2^* is of even length. In particular each $H_1 \,\square_w\, H_2$ admits a NZ 3-flow and hence, by (II), it follows that $G_1 \,\square\, G_2$ also admits a NZ 3-flow. ∎

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Received
Accepted