Tree-like isometric subgraphs of hypercubes

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Abstract

Tree-like isometric subgraphs of hypercubes, or tree-like partial cubes as we shall call them, are a generalization of median graphs. Just as median graphs they capture numerous properties of trees, but encompass a larger class of graphs that may be easier to recognize than the class of median graphs. We investigate the structure of tree-like partial cubes, characterize them, and provide examples of similarities with trees and median graphs. For instance, we show that every tree-like partial cube G contains a cube that is invariant under every automorphism of G. This property is shared with median graphs and generalizes the existence of a center in trees. Furthermore, we establish a connection with dismantlable graphs, and show that every finite group is isomorphic to the automorphism group of a tree-like partial cube and, more restrictively, to that of a median graph.

Key words: isometric embeddings, partial cubes, expansion procedures, trees, median graphs, graph automorphisms, automorphism groups, dismantlable graphs.

1 Introduction

By a *partial cube* we mean an isometric subgraph of a hypercube. Perhaps the most important subclass of partial cubes is the class of median graphs. They possess rich structure and have many interesting properties, cf. [15]. There is also an injection of the class of triangle-free graphs into the class of median graphs of diameter 4, see [13].

Unfortunately, just a small portion of the properties of median graphs extends to partial cubes. For instance, every median graph contains a cube that is invariant under all automorphisms of G [3]. Clearly, this result is not generally true in the class of partial cubes, the simplest example being C_6. Also, regular median graphs are precisely hypercubes [17], whereas the class of regular partial cubes seems to be rather rich, cf. [14]. Thus one can pose the following question: Is there a natural class of graphs between median graphs and partial cubes that captures "most" of the properties of median graphs?

Our main purpose is to introduce such a class of graphs, we call them tree-like partial cubes. In Section 3 we characterize them as the graphs containing no gated periphery-free subgraph. This in particular implies that one can use *any* sequence of peripheral contractions to obtain K_1 from a tree-like partial cube. We follow with a section in which we list several properties that are shared by median graphs and tree-like partial cubes. In particular we show that hypercubes are the only regular tree-like partial cubes, and that the cube graph of a tree-like partial cube is dismantlable. Then, in Section 5, we consider automorphisms of tree-like partial cubes. We prove that any such graph G contains a cube that is invariant under every automorphism of G and that every finite group is isomorphic to the automorphism group of a tree-like partial cube and of a median graph. Finally, let us add that we obtain several new and shorter proofs of results on median graphs as a by-product.

2 Notation and preliminaries

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k; for short also called *cube.* A graph G is called *prime* (with respect to the Cartesian product) if it cannot be represented as the product of two nontrivial graphs, that is, if $G = G_1 \Box G_2$ implies that G_1 or G_2 is the one-vertex graph K_1.

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the length of a shortest u, v-path. A subgraph H of a graph G is *convex* if for any two vertices u, v of H all shortest paths between u and v in G are already in H. A subgraph H of a graph G is called *gated* in G if for every $x \in V(G)$

there exists a vertex u in H such that u lies on a shortest x, v-path for all $v \in V(H)$. Clearly gated subgraphs are convex and convex subgraphs isometric.

Isometric subgraphs of hypercubes are called *partial cubes*. By the above, convex and gated subgraphs of partial cubes are also partial cubes.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u, v-path, a shortest u, w-path, and a shortest w, v-path. Median graphs are partial cubes, cf. [18, 12].

Two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler [9, 26] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Winkler [26] proved that a bipartite graph is a partial cube if and only if Θ is transitive.

Let $G = (V, E)$ be a connected, bipartite graph and ab an edge of G. Then the following sets are of relevance for partial cubes:

$$W_{ab} = \{w \in V \mid d_G(a, w) < d_G(b, w)\},$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\},$$
$$F_{ab} = \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.$$

Clearly, W_{ab} and W_{ba} are disjoint and $V = W_{ab} \cup W_{ba}$ because G is bipartite.

Let $G = (V, E)$ be a graph, V_1 and V_2 subsets of V with nonempty intersection, and $V = V_1 \cup V_2$. Suppose that $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are isometric in G and that no vertex of $V_1 \setminus V_2$ is adjacent to a vertex of $V_2 \setminus V_1$. ($\langle W \rangle$ stands for the subgraph induced by W.) Then the *expansion* of a graph G with respect to V_1 and V_2 is the graph obtained from G by the following procedure:

(i) Replacement of each vertex $v \in V_1 \cap V_2$ by vertices v_1, v_2 and insertion of the edge $v_1 v_2$.

(ii) Insertion of edges between v_1 and the neighbors of v in $V_1 \setminus V_2$ as well as between v_2 and the neighbors of v in $V_2 \setminus V_1$.

(iii) Insertion of the edges $v_1 u_1$ and $v_2 u_2$ whenever $v, u \in V_1 \cap V_2$ are adjacent in G.

If $V_1 \cap V_2$ is convex in G, we speak of a *convex expansion*, if $V_1 \cap V_2$ is isometric in G, then the expansion is called *isometric*. *Contraction* is the operation inverse to the expansion. Partial cubes were characterized as graphs that can be obtained from K_1 by a sequence of expansions [7].

An expansion is called *peripheral* if at least one of the covering sets V_1 or V_2 is equal to $V(G)$. Note that then the other set equals the intersection, which is thus necessarily isometric in G. Now, a graph is called a *tree-like partial cube* if it can be obtained by a sequence of peripheral expansions from K_1. Finally, a subset U_{ab} is called a *periphery* if $U_{ab} = W_{ab}$. The corresponding Θ-class F_{ab} will be called *periphery inducing*.

3 Characterization of tree-like partial cubes

Median graphs can be characterized as graphs that can be obtained from K_1 by a sequence of convex expansions [16, 18]. Moreover, by [19, Lemma 9], these expansions can be assumed to be peripheral. Hence, by definition, tree-like partial cubes extend this important property of median graphs.

It is obvious that every tree-like partial cube has a periphery, which is not true for partial cubes in general. For instance, the even cycles C_{2n}, $n \geq 3$, or the graph of Fig. 1 are partial cubes without a periphery. We call such graphs *periphery-free* partial cubes. Another example of periphery-free partial cubes are graphs obtained from complete graphs by subdivision of every edge, see [14].

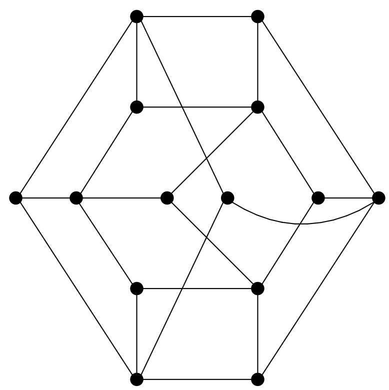

Figure 1: A periphery-free partial cube

We will characterize tree-like partial cubes via periphery-free subgraphs. It is easy to prove the following proposition.

Proposition 3.1 *Let G be a periphery-free partial cube, and H be obtained from G by an expansion. Then H is periphery-free if and only if the expansion is not peripheral.*

We can then characterize the new class of graphs as follows:

Theorem 3.2 *A partial cube is tree-like if and only if it contains no gated periphery-free subgraph.*

Proof. Let G be a smallest tree-like partial cube that contains a gated periphery-free subgraph. So, let H be a periphery-free graph isomorphic to a gated subgraph G_1 of G, and G be obtained by a peripheral expansion from a tree-like partial cube G'. By minimality, G' contains no gated periphery-free subgraph.

Suppose that F_{ab} is a Θ-class of G that intersects the edge-set of G_1. We claim that F_{ab} is not periphery inducing. Let $xy \in F_{ab} \cap E(G_1)$. Since H is periphery-free there exists a vertex $u \in V(H)$ that is in W_{xy} but not in U_{xy} with respect to H. By the convexity of G_1 in G, the vertex corresponding to u is also not in U_{xy} with respect to G, hence F_{xy} is not periphery inducing in G.

Therefore G_1 is contained either in a subgraph of G corresponding to G' or in $G \setminus G'$. In the first case we infer from the isometry of G' that G_1 is also gated in G', in contradiction to the minimality assumption. In the second case we consider the subgraph G_2 of G' isomorphic to G_1, induced by vertices that are matched via the periphery inducing Θ-class. We claim that G_2 is gated in G'. Indeed, the distance from any vertex x of G' to a vertex of G_1 is exactly 1 plus the distance from x to the corresponding vertex of G_2. Hence the gatedness of G_2 clearly follows from the gatedness of G_1. Thus G_2 is a periphery-free subgraph of G', again in contradiction to the minimality assumption.

For the converse assume that G is a partial cube that contains no gated periphery-free subgraphs. Then G is not periphery-free and one can obtain G by a peripheral expansion from a graph G'. If G' would contain a gated periphery-free subgraph G_1 then G_1 would be gated also in G (we use the same arguments as above by considering the distances of vertices from $G \setminus G'$ to vertices of G'). By induction on the number of vertices we infer that G' is a tree-like partial cube, and thus also G. □

Corollary 3.3 *For any periphery U of a tree-like partial cube G, $G \setminus U$ is a tree-like partial cube.*

This means that one can use any sequence of peripheral contractions to obtain K_1 from a tree-like partial cube. Note that this is a generalization of the elimination procedure in trees where pendant vertices in trees are contracted (or removed). This justifies the name "tree-like" partial cubes.

Corollary 3.4 *$G \Box H$ is a tree-like partial cube if and only if G and H are tree-like partial cubes.*

Proof. G and H are isometric (in fact, even convex) subgraphs of $G \Box H$. Hence $G \Box H$ is a partial cube if and only if G and H are partial cubes. Moreover, the Θ-classes of $G \Box H$ naturally correspond to the Θ-classes of G and H, cf. [12, Lemma 4.3]. Therefore, if G and H are tree-like, then so is $G \Box H$. For the converse we use induction on $|V(G)| + |V(H)|$ combined with Corollary 3.3. □

4 Properties of median graphs shared by tree-like partial cubes

By definition tree-like partial cubes can be obtained from K_1 by a sequence of peripheral expansions, just as median graphs can be obtained from K_1 by a sequence of peripheral (convex) expansions [19]. In this section we list several additional properties that can be extended from median graphs to tree-like partial cubes. We begin with the characterization of regular tree-like partial cubes.

Theorem 4.1 *Regular tree-like partial cubes are hypercubes.*

Proof. Let G be a regular tree-like partial cube and suppose that it is not a hypercube. If G is isomorphic to $K_2 \Box U$ for some peripheral subgraph U, then by Corollary 3.4, U is also a regular tree-like partial cube that is not a hypercube, and induction completes the proof. On the other hand, if for some peripheral subgraph U, G is not isomorphic to $K_2 \Box U$, then $K_2 \Box U$ is a proper induced subgraph of G. But then G clearly cannot be regular. □

Since Cartesian products of regular partial cubes are regular partial cubes, the problem of characterizing regular partial cubes reduces to partial cubes that are prime with respect to the Cartesian product. By the same idea as above we can easily prove the following corollary:

Corollary 4.2 *Let G be a regular, prime (with respect to the Cartesian product) partial cube on at least three vertices. Then G is periphery-free.*

Intersection graphs of maximal hypercubes will be briefly called *cube graphs*. Thus H is the cube graph of a graph G, in symbols $H = Q(G)$, when the vertices of H are the maximal hypercubes of G, two vertices in H being adjacent whenever the corresponding hypercubes in G intersect. It was observed by Bandelt and van de Vel [4] that the cube graph of a median graph is always *Helly*, that is a graph in that balls have the Helly property. We cannot extend this property to tree-like partial cubes, as the example in Fig. 2 shows.

On the other hand, Helly graphs belong to the class of dismantlable graphs that are defined by an elimination procedure, that is a generalization of the elimination of simplicial vertices in chordal graphs. We say that a vertex u in a graph G is *d-simplicial* if there exists a neighbor v of u such that all neighbors of u except v are also neighbors of v. If G can be reduced to the one-vertex graph by successive removal of d-simplicial vertices then G is called a *dismantlable graph*. Dismantlable graphs were studied in [21] under the name *cop-win graphs*, see also [8]. Below we show that the cube graph of a tree-like partial cube is dismantlable.

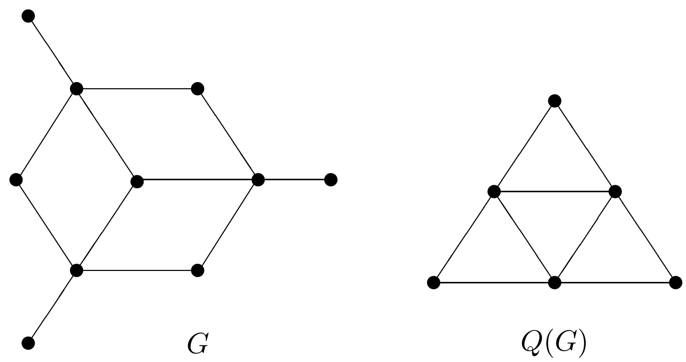

Figure 2: A tree-like partial cube and its cube graph

Let G^{Δ} denote the graph obtained from a graph G that has the same vertex set as G and in that two vertices are adjacent whenever they are in the same hypercube of G (this operation is from [4]).

Proposition 4.3 *For any tree-like partial cube G, the graphs G^{Δ} and $Q(G)$ are dismantlable.*

Proof. To see that G^{Δ} is dismantlable, let U be the subgraph of G obtained in the last expansion step. Then each vertex u in U has a unique neighbor u' in $G \setminus U$. It is clear that every maximal hypercube that includes u includes also u', thus u is d-simplicial in G^{Δ}. By removing all d-simplicial vertices of U one by one we obtain the graph $G \setminus U$ that is a smaller tree-like partial cube than G, and we can conclude the argument by induction (of course, $K_1^{\Delta} = K_1$, which is dismantlable by definition).

Note that the cube graph of a partial cube G coincides with the clique graph of G^{Δ} (where the *clique graph* of a graph is the intersection graph of maximal complete subgraphs of a graph). Bandelt and Prisner [2] showed that dismantlable graphs are invariant under the clique graph transformation, that is, the clique graph of a dismantlable graph is again dismantlable. Combining the above two observations we conclude that the cube graph of a tree-like partial cube is dismantlable. □

Cube graphs of all standard examples of partial cubes that are not tree-like are not dismantlable. Therefore we ask the following question: *Is a partial cube tree-like precisely when its cube graph is dismantlable?*

Yet another interesting feature of median graphs are tree-like equalities. Perhaps the most interesting one is the following. Let α_i be the number of induced i-cubes in a median graph, then

$$\sum_{i \geq 0} (-1)^i \, \alpha_i = 1\,.$$

This result was discovered by Soltan and Chepoi [24] and independently by Škrekovski [23], see also [6] for further generalizations. It is easy to prove that the above equality extends to tree-like partial cubes. We leave the proof to the reader. (Note that for the graph of Fig. 1 the corresponding sum is 2.)

5 Automorphisms of tree-like partial cubes

Recall that any tree T contains either a vertex or an edge that is invariant under every automorphism of T, cf. [20]. This property extends to median graphs because automorphisms of a median graph always fix a cube [3]. We next extend this result to tree-like partial cubes.

Theorem 5.1 *Let G be a tree-like partial cube. Then G contains a cube that is invariant under every automorphism of G.*

Proof. Let G be a smallest counterexample to the theorem and $U_1, \ldots, U_k$ the peripheries of G. Consider

$$I = \bigcap_{i=1}^{k} G \setminus U_i.$$

Suppose I is nonempty. We claim that I is a tree-like partial cube. Suppose that I contains a gated periphery-free subgraph H. In view of Theorem 3.2 we wish to prove that then H is also gated in G. Let x be a vertex of U_i for which there is no gate in H and consider the vertex x' in $G \setminus U_i$ that corresponds to x. Note that the distance from x' to a vertex of H is exactly 1 less than the distance from x to the same vertex of H. Thus x' also has no gate in H. If x' is in some periphery U_j we use the same argument again. Note that x' lies in one periphery less than x. We repeat this until we arrive at a vertex of I that has no gate in H, a contradiction.

Hence I is a tree-like partial cube. Thus, by the minimality assumption, it contains a cube C that is invariant under every automorphism of I. Now, observe that the collection of peripheries is invariant under every automorphism of G, and thus also I. Therefore, every automorphism of G preserves the cube C.

Now, suppose that I is empty. We claim that this is the case if and only if there is a U_i with $|U_i| = |V(G)|/2$. Clearly the condition is sufficient. To show that it is necessary, suppose that $|U_i| < |V(G)|/2$ for all i. Hence $|V(G \setminus U_i)| > |V(G)|/2$ for all i, thus the subgraphs $G \setminus U_i$ are mutually nondisjoint (gated) subgraphs of G. As gated subgraphs always enjoy the Helly property, cf. [25], the common intersection of these subgraphs I is also nonempty.

Hence if $I = \bigcap_{i=1}^{k}(G \setminus U_i)$ is empty then $G \cong U_i \Box K_2$ for some i. Since U_i is smaller than G we infer that U_i contains a cube that is preserved under every automorphism of U_i. We describe this cube in more detail. Let $U_i = U \Box K_2^s$, where U and K_2^s are relatively prime. By our assumptions U is tree-like and has a cube C that is preserved by every automorphism of U_i because the automorphism group of the Cartesian product of relatively prime graphs is the direct product of the groups of the factors, see [12, Corollary 4.17]. By the same argument $C \Box K_2^{s+1}$ is a cube that is preserved by every automorphism of G. □

Automorphisms of periphery-free partial cubes need not preserve a cube. One can easily find such automorphisms on the graph of Fig. 1, on even cycles of length at least six, and on subdivisions of complete graphs (the corresponding automorphisms for the latter graphs are derived from the derangement induced automorphisms of complete graphs). A natural question arises: *Let G be a periphery-free partial cube that is minimal with respect to the expansion sequence. Does there exist an automorphism that does not preserve any cube of G?*

Another property of tree-like partial cubes is the following:

Proposition 5.2 *Every finite (abstract) group is isomorphic to the automorphism group of some tree-like partial cube.*

Proof. Let A be a finite group. Then, by Frucht's theorem [10], A is isomorphic to the automorphism group $\mathrm{Aut}(G)$ of some graph G. If G is a tree, we are done. Otherwise, let $f(G)$ be the graph constructed as follows. Subdivide each edge of G, add a new vertex z and connect all original vertices of G to z. In the graph $f(G)$, let the vertices at level 1 and 2, be the vertices of distance 1, resp. 2, from z. Thus, the vertices at level 1 correspond to vertices of G, and vertices at level 2 have degree 2.

We now label the vertices of $f(G)$ as follows. First label the vertex z with $n = |V(G)|$ zeros. Then label its neighbors with strings consisting of $n-1$ zeros and with one 1, each in a different position. Finally, vertices at level 2 receive $n-2$ zeros and two 1's in the same positions as their two neighbors. It is clear that $f(G)$ isometrically embeds into Q_n. Hence $f(G)$ is a partial cube. In addition, every Θ-class of $f(G)$ is periphery inducing. (Each periphery is isomorphic to $K_{1,r}$, where r is the degree of the corresponding vertex in G). We conclude that $f(G)$ is a tree-like partial cube.

Finally note (cf. [13]) that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(f(G))$ and so A is isomorphic to $\mathrm{Aut}(f(G))$. □

In [13] and [12] it has also been shown that $f(G)$ is a median graph if and only if G is triangle free. Since $f(G)$ is triangle-free, we infer that $f(f(G))$ is a median graph. We therefore have the following corollary:

Corollary 5.3 *Every finite (abstract) group is isomorphic to the automorphism group of a median graph.*

We wish to add that Sabidussi [22] showed that one can impose additional conditions in Frucht's theorem, in particular arbitrary chromatic number. Since two-chromatic graphs are bipartite and since bipartite graphs are triangle-free, there exists a triangle-free graph H to every group A such that A is isomorphic to $\mathrm{Aut}(H)$. Our graph $f(G)$ is a special case of Sabidussi's construction.

Recall [1] that a class of graphs $\mathcal{C}$ is *isomorphism complete* if the graph isomorphism problem can be reduced to isomorphism within $\mathcal{C}$. The previous proof allows us to add tree-like partial cubes and median graphs to the list of such classes, cf. [1, p. 1514]:

Corollary 5.4 *The classes of tree-like partial cubes and median graphs are isomorphism complete.*

Proof. For a graph G let $f(G)$ be the tree-like partial cube as defined in the proof of Proposition 5.2. Then the observation that two graphs G and H are isomorphic if and only if $f(G)$ and $f(H)$ are isomorphic yields the first assertion.

For the second one it suffices to restrict attention to triangle-free graphs G. $\square$

We conclude with a remark about the hierarchy of partial cubes as introduced in [11] and refined in [5]. According to that classification, tree-like partial cubes belong to the class of graphs obtainable by an isometric expansion procedure and are thus semi-median graphs. However, they are not almost-median in general. To see this, consider the graph obtained by peripherally expanding the (isometric) 6-cycle of the vertex-deleted 3-cube. For an example of an almost-median periphery-free partial cube, see the graph of Fig. 1.

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