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A NOTE ON THE UNIQUE FACTORIZATION THEOREM FOR PROPERTIES OF INFINITE GRAPHS

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Abstract. A property $\mathcal{P}$ of infinite graphs is said to be of *finite character* if a graph G has property $\mathcal{P}$ if and only if every finite vertex-induced subgraph of G has property $\mathcal{P}$. Using a generalization of the well-known Erdős-de Bruijn Theorem for arbitrary properties of finite character, we present a proof of the Unique Factorization Theorem for additive properties of infinite graphs which are of finite character.

1. Compactness Theorem for Generalized colouring

Let $\mathcal{I}^*$ be the class of all simple graphs (finite or infinite), a graph property $\mathcal{P}$ is any isomorphism-closed nonempty proper subclass of $\mathcal{I}^*$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring of a graph $G = \langle V, E\rangle$ is a partition $(V_1, V_2, \ldots, V_n)$ of V such that each partition class V_i induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$. For convenience, we allow empty partition classes in the partition sequence. Such a partition induces the null graph $K_0 = \langle \emptyset, \emptyset\rangle$. If each of the $\mathcal{P}_i$'s, $i = 1, 2, \ldots, n$, is the property $\mathcal{O}$ of being edgeless, we have a regular n-colouring. Many other examples, references and results on generalized colourings of finite graphs may be found e.g. in [1, 2].

In 1951, de Bruijn and Erdős proved that an infinite graph G is k-colourable if and only if every finite subgraph of G is k-colourable. An analogous compactness theorem for generalized colourings was proved in [3].

The key concept for the Vertex Colouring Compactness Theorem of [3] is that of a property being of *finite character*. Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of *finite character* if a graph in $\mathcal{I}^*$ has property $\mathcal{P}$ if and only if each finite vertex-induced subgraph has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character; for example the graph property $\mathcal{Q}$ of not containing a vertex of infinite degree

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is induced-hereditary but not of finite vertex character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply *hereditary*) is induced-hereditary as well. The properties of being edgeless, of maximum degree at most k, K_n-free, acyclic, complete, perfect, etc. are properties of finite character. The compactness theorem for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourings, where the $\mathcal{P}_i$'s are of finite character, have been proved using Rado's Selection Lemma in [3]:

Theorem 1.1 (Vertex Colouring Compactness Theorem-VCCT)**.** [3] *Let G be a graph in $\mathcal{I}^*$ and let $\mathcal{P}_1, \mathcal{P}_2 \ldots \mathcal{P}_n$ be properties of graphs of finite character. Then G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable if every finite induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable.*

Let us remark that an analogous result is presented in the Ph.D. thesis [4], where M. Dorfling proved the Compactness Theorem for generalized colourings using the classical Tychonoff topological Compactness Theorem.

2. Reducibility and Unique Factorization Theorem

Given a positive integer n and graph properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. By the property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ we understand the class of all graphs that are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable. The binary operation $\circ$ is obviously commutative, associative on the class of graph properties with the neutral element $\Theta = \{K_0\}$. A graph property $\mathcal{P}$ is said to be *reducible* if there exist graph properties $\mathcal{P}_1, \mathcal{P}_2 \neq \Theta$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$ and *irreducible* otherwise. In general, there are only few graph properties that have a unique factorization into irreducible ones (see [5, 7]). The problem of unique factorization of a reducible induced-hereditary property into induced-hereditary factors was introduced in connection with the study of the existence of uniquely colourable graphs with respect to hereditary properties (see [2, 1] and Problem 17.9. in the book [10]). In [12] it is proved that every reducible property of finite graphs, which is closed under taking subgraphs and disjoint union of graphs (such properties are called *hereditary and additive*), is uniquely factorizable into irreducible hereditary additive factors. An analogous result was obtained in [11, 6] for additive induced-hereditary properties of finite graphs (following [1] we denote by $\mathbb{M}^a$ the set of all additive induced-hereditary properties of finite graphs):

Theorem 2.1 (Unique Factorization Theorem - UFT)**.** [11, 6] *Every additive induced-hereditary property of finite graphs is in $\mathbb{M}^a$ uniquely factorizable into a finite number of irreducible additive induced-hereditary factors (up to the order of factors).*

The aim of this note is to show that the Vertex Colouring Compactness Theorem and the Unique Factorization Theorem for additive induced-hereditary properties of finite graphs imply Unique Factorization for additive properties of finite character.

Let α be an infinite cardinal. Let us denote by $\mathcal{I}(\alpha)$ the class of all simple graphs G having order less than or equal to α. The symbol $\mathbb{M}^a(\alpha)$ will denote the set of all additive induced-hereditary properties of graphs belonging to $\mathcal{I}(\alpha)$. In [9] it is proved that $(\mathbb{M}^a(\alpha), \subseteq)$ is a lattice, which is completely distributive and it is remarked that the same results can be obtained for partially ordered sets having the cardinality less than or equal to α. The next result states that the properties from $\mathbb{M}^a(\alpha)$ that are of finite character form a sublattice of $\mathbb{M}^a(\alpha)$. In addition this sublattice is isomorphic to $\mathbb{M}^a$.

Theorem 2.2. *Let $\mathbb{M}^{afc}(\alpha)$ be the set of all properties of finite character from $\mathbb{M}^a(\alpha)$. Then $(\mathbb{M}^{afc}(\alpha), \subseteq)$ is a completely distributive lattice. Moreover $(\mathbb{M}^{afc}(\alpha), \subseteq)$ is isomorphic to the lattice $\mathbb{M}^a$ of additive induced-hereditary properties of finite graphs.*

Proof. The proof of the fact that $(\mathbb{M}^{afc}(\alpha), \subseteq)$ is a completely distributive lattice is analohous to that in [9] and therefore omitted.

In order to prove the second assertion of the theorem, we utilize the idea presented by M. Dorfling in [4]. Given an additive induced-hereditary property $\mathcal{P}$, denote by $\mathcal{P}^*(\alpha)$ the class of graphs belonging to $\mathcal{I}(\alpha)$ such that $G \in \mathcal{P}^*(\alpha)$ if and only if each finite induced-subgraph of G is in $\mathcal{P} \in \mathbb{M}^a$. According to the definition $\mathcal{P}^*(\alpha)$ is of finite character and closed under disjoint union, so that $\mathcal{P}^*(\alpha) \in \mathbb{M}^{afc}(\alpha)$. We prove that the mapping $\Phi : \mathbb{M}^a \to \mathbb{M}^{afc}(\alpha)$, defined by $\Phi(\mathcal{P}) = \mathcal{P}^*(\alpha)$, is an isomorphism of the lattices $\mathbb{M}^a$ and $\mathbb{M}^{afc}(\alpha)$.

Indeed, if $\mathcal{P}_1, \mathcal{P}_2$ are two distinct properties from $\mathbb{M}^a$ then without loss of generality we can assume that $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$. Then there exists a graph $G \in \mathcal{P}_2 \setminus \mathcal{P}_1$. Then obviously G belongs to $\Phi(\mathcal{P}_2) \setminus \Phi(\mathcal{P}_1)$ and therefore $\Phi(\mathcal{P}_1) = \mathcal{P}_1^*(\alpha) \neq \mathcal{P}_2^*(\alpha) = \Phi(\mathcal{P}_2)$. Suppose now, that $\mathcal{Q}(\alpha)$ is a property from $\mathbb{M}^{afc}(\alpha)$. Then we define the property

$$\mathcal{Q} = \{G : G \text{ is a finite induced subgraph of some } H \in \mathcal{Q}(\alpha)\}.$$

One can easily see that $\mathcal{Q}$ belongs to $\mathbb{M}^a$ and moreover $\Phi(\mathcal{Q}) = \mathcal{Q}(\alpha)$.

It remains to show that Φ preserves the partial order $\subseteq$. Let us consider two properties $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and their images $\Phi(\mathcal{P}_1) = \mathcal{P}_1^*(\alpha)$ and $\Phi(\mathcal{P}_2) = \mathcal{P}_2^*(\alpha)$. Hence, if $G \in \mathcal{P}_1^*(\alpha)$ then, according to the definition of Φ, each finite induced subgraph of G belongs to $\mathcal{P}_1$. But $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and it implies that each finite induced subgraph of G belongs to $\mathcal{P}_2$. Therefore $G \in \mathcal{P}_2^*(\alpha)$ and we have $\mathcal{P}_1^*(\alpha) \subseteq \mathcal{P}_2^*(\alpha)$ that means that Φ preserves partial order $\subseteq$. $\square$

Now we are ready to prove our main result:

Theorem 2.3. *Every additive property of finite character is uniquely factorizable into finite number of irreducible factors belonging to $\mathbb{M}^{afc}(\alpha)$.*

Proof. Using VCCT we have that an infinite graph $G \in \mathcal{I}(\alpha)$ has property $\mathcal{P}_1(\alpha) \circ \mathcal{P}_2(\alpha) \circ \cdots \circ \mathcal{P}_n(\alpha)$ if and only if every finite induced subgraph of G has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ in $\mathbb{M}^a$. It implies that $\mathbb{M}^{afc}(\alpha)$ is closed under the product operation.

Suppose now, that a property $\mathcal{P}(\alpha) \in \mathcal{I}(\alpha)$ has two distinct factorizations in $\mathbb{M}^{afc}(\alpha)$, say $\mathcal{P}_1(\alpha) \circ \mathcal{P}_2(\alpha) \circ \cdots \circ \mathcal{P}_n(\alpha)$ and $\mathcal{Q}_1(\alpha) \circ \mathcal{Q}_2(\alpha) \circ \cdots \circ \mathcal{Q}_k(\alpha)$. Let G

be a graph belonging to $\mathcal{P}(\alpha)$. Then each its finite induced subgraph has a $(\mathcal{P}_1(\alpha), \mathcal{P}_2(\alpha), \ldots, \mathcal{P}_n(\alpha))$-factorization and also a $(\mathcal{Q}_1(\alpha), \mathcal{Q}_2(\alpha), \ldots, \mathcal{Q}_n(\alpha))$-factorization. Thus, according to Theorem 2.2, the property $\Phi^{-1}(\mathcal{P}(\alpha)) \in \mathbb{M}^a$ (where Φ is defined as in the proof of the previous theorem) has two factorizations, namely $\Phi^{-1}(\mathcal{P}_1(\alpha))\circ \cdots \circ\Phi^{-1}(\mathcal{P}_n(\alpha))$, and $\Phi^{-1}(\mathcal{Q}_1(\alpha))\circ \cdots \circ\Phi^{-1}(\mathcal{Q}_k(\alpha))$ into irreducible factors from $\mathbb{M}^a$. But it contradicts UFT. Therefore the factorization of $\mathcal{P}(\alpha)$ in $\mathbb{M}^{afc}$ must be unique and the proof is complete. □

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