Reconstructing subgraph-counting graph polynomials of increasing families of graphs [1]

Boštjan Brešar

University of Maribor, FERI, Smetanova 17, 2000 Maribor, Slovenia
e-mail: bostjan.bresar@uni-mb.si

Wilfried Imrich

Montanuniversität Leoben, A-8700 Leoben, Austria
e-mail: imrich@unileoben.ac.at

Sandi Klavžar

Department of Mathematics and Computer Science
University of Maribor, PeF
Koroška cesta 160, 2000 Maribor, Slovenia
e-mail: sandi.klavzar@uni-mb.si

Abstract

A graph polynomial $P(G, x)$ is called reconstructible if it is uniquely determined by the polynomials of the vertex deleted subgraphs of G for every graph G with at least three vertices. In this note it is shown that subgraph-counting graph polynomials of increasing families of graphs are reconstructible if and only if each graph from the corresponding defining family is reconstructible from its polynomial deck. In particular we prove that the cube polynomial is reconstructible. Other reconstructible polynomials are the clique, the path and the independence polynomial. Along the way two new characterizations of hypercubes are obtained.

Keywords: graph polynomial; hypercube; reconstruction; cube polynomial; increasing family of graphs

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1 Introduction

Let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let $G_i = G - v_i$, $1 \le i \le n$, be its vertex deleted subgraphs. Then the multiset $\{G_1, G_2, \ldots, G_n\}$ is called the *deck* of G. A graph G is called *reconstructible* if it is uniquely determined (up to isomorphism) by its deck. The notorious *reconstruction conjecture* (also known as Kelly-Ulam conjecture) asserts that all finite graphs on at least 3 vertices are reconstructible, cf. [1].

More generally, a *graph property is reconstructible* if it is uniquely determined by the deck of a graph. Many graph properties have been proved to be reconstructible, for instance, the number of hamiltonian cycles and the number of one-factors, cf. [18] for these and many more properties. In addition, Tutte [26] proved that the characteristic polynomial, the chromatic polynomial, and their generalizations are also reconstructible, as well as the matching polynomial [9,13]. For more information on the reconstruction of classical graph polynomials see the survey [7]. In the same paper Farrell also observed that the reconstruction conjecture can be stated in terms of reconstructible graph polynomials.

Given a graph property, do we really need the deck of a graph for its reconstruction? In particular, given a graph polynomial, can it be reconstructed from its *polynomial deck*, that is, from the multiset of the polynomials of the vertex deleted subgraphs? For the characteristic polynomial, Gutman and Cvetković [15] posed this question in 1975, but the problem remains open. Recently, Hagos [16] proved that the characteristic polynomial of a graph is reconstructible from the polynomial deck of a graph together with the polynomial deck of its complement. For related results see [25] and [24]; in the latter paper Schwenk suspects that the answer to the question is negative.

In this paper we consider the problem of reconstructing a graph polynomial from its polynomial deck for a class of polynomials that are defined as generating functions for the numbers of subgraphs from given increasing families of graphs. These subgraph counting polynomials are instances of F-polynomials in the sense of Farrell [6].

In the next section we formally introduce these polynomials and prove that such polynomials are reconstructible from the polynomial deck if and only if each graph from the corresponding defining family is reconstructible from its polynomial deck. The well known clique, independence, star and path polynomial are of this type as well as the recently introduced cube polynomial [3]. In Section 3 we prove that the cube polynomial is also reconstructible. Two related characterizations of hypercubes are given, for example, a graph is a hypercube if and only if its cube polynomial is of the form $(x + 2)^k$.

2 Reconstruction of $\mathcal{H}$-polynomials

Let $\mathcal{H} = \{H_0, H_1, H_2, \ldots\}$ be a family of graphs such that $H_0 = K_1$ and H_i is an induced subgraph of H_{i+1} for $i = 0, 1, 2, \ldots$. We call such a family of graphs an *increasing family*. Given an increasing family $\mathcal{H}$, and an arbitrary graph G, we denote by $p_i(G)$ the number of induced H_i's in G. The *$\mathcal{H}$-polynomial* $P_{\mathcal{H}}(G, x)$ of a graph G is the generating function for the $p_i(G)$, that is,

$$P_{\mathcal{H}}(G, x) = \sum_{i \geq 0} p_i(G) x^i \,. \tag{1}$$

For example, setting $H_i = K_i$, respectively $H_i = \overline{K_i}$ or $H_i = Q_i$, one obtains the *clique polynomial* [10,17], the *independence polynomial* [4,14,17], and the *cube polynomial* [3]. Definition (1) is often stated in a slightly different form as

$$P_{\mathcal{H}}(G, x) = 1 + \sum_{i \geq 1} p_{i-1}(G) x^i \,,$$

or even as

$$P_{\mathcal{H}}(G, x) = 1 + \sum_{i \geq 1} (-1)^i p_{i-1}(G) x^i \,,$$

but for our purposes any one of these definitions is practicable, so we will adhere to (1).

Remark 1 *Let $\mathcal{H} = \{H_0, H_1, \ldots\}$ be an increasing family of graphs, and P the corresponding $\mathcal{H}$-polynomial. Then each element of $\mathcal{H}$ is characterized by the polynomial P in the sense of Farrell [8]. Indeed, suppose that G is a graph such that $P(G, x) = P(H_j, x)$. Then G should contain H_j as an induced subgraph. Since $|V(G)| = |V(H_j)|$ this is only possible when G and H_j are isomorphic.*

Let $\mathcal{H}$ be an increasing family of graphs. We say that an *$\mathcal{H}$-polynomial is reconstructible from the polynomial deck*, if for every graph G on at least three vertices, the multiset $\{P_{\mathcal{H}}(G-v, x); v \in V(G)\}$ uniquely determines $P_{\mathcal{H}}(G, x)$. Furthermore, we say that a *graph G is reconstructible from the polynomial deck* if the multiset $\{P_{\mathcal{H}}(G - v, x); v \in V(G)\}$ uniquely determines G (up to isomorphism).

Since a graph on two vertices is not reconstructible (in the usual sense), it is also not reconstructible from its polynomial deck. Nevertheless we wish to observe that our definitions allow that H_1 is a graph on two vertices. In fact, it is a K_2 for all polynomials considered here with the exception of the independence polynomial. In that case $H_1 = \overline{K_2}$.

In the following theorem we will make use of Kelly's lemma, cf. [12, p. 62, Lemma 4.5.1]. By $p_H(G)$ we denote the number of induced subgraphs of a graph G that are isomorphic to H.

Lemma 2 *(Kelly's lemma) Let H be an arbitrary graph on m vertices, G is a graph on n $(n > m)$ vertices, and $G_i, i = 1, \ldots, n$ its vertex deleted subgraphs. Then*

$$(n-m)p_H(G) = \sum_{i=1}^{n} p_H(G_i).$$

The same formula holds if $p_H(G)$ denotes the number of (induced or non induced) subgraphs of G that are isomorphic to H. The restriction to induced subgraphs is important in the following theorem.

Theorem 3 *Let $\mathcal{H} = \{H_0, H_1, \ldots\}$ be an increasing family of graphs, and $P_{\mathcal{H}}$ the corresponding $\mathcal{H}$-polynomial. Then $P_{\mathcal{H}}$ is reconstructible from the polynomial deck if and only if each H_j (where $|V(H_j)| \geq 3$) is reconstructible from its polynomial deck.*

Proof. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}, n \geq 3$, and $G_i = G - v_i$, $1 \leq i \leq n$.

Suppose first that each $H_j, j = 1, 2 \ldots,$ on at least three vertices is reconstructible from its polynomial deck. For $|V(H_j)| < n$ Kelly's lemma implies

$$p_j(G) = \frac{\sum_{i=1}^{n} p_j(G_i)}{n - |V(H_j)|}\,. \tag{2}$$

Let ℓ be the largest index such that $|V(H_\ell)| \leq n$. If $|V(H_\ell)| < n$ then $P(G, x)$ can be reconstructed using (2). Now suppose that $|V(H_\ell)| = n$.

If $\{P(G_i, x), i = 1, \ldots, n\}$ is different from the polynomial deck of H_ℓ, then $G \neq H_\ell$. Since G and H_ℓ have the same number of vertices, H_ℓ cannot be an induced subgraph of G. Hence each $p_j(G)$ can be reconstructed using (2), and so $P(G, x)$ is determined.

On the other hand, if $\{P(G_i, x), i = 1, \ldots, n\}$ coincides with the polynomial deck of H_ℓ, then G is isomorphic to H_ℓ by the assumptions of the theorem. Hence $P(G, x) = P(H_\ell, x)$.

For the converse suppose that G is not isomorphic to H_ℓ, but that the polynomial deck of H_ℓ coincides with $\{P(G_i, x), i = 1, \ldots, n\}$ for some ℓ. Then $P(H_\ell, x)$ is of degree ℓ, whereas $P(G, x)$ is of degree at most $\ell - 1$, a contradiction. □

Let

$\mathcal{K} = \{K_1, K_2, K_3, \ldots\}$ be the family of complete graphs (cliques),
$\mathcal{I} = \{\overline{K_1}, \overline{K_2}, \overline{K_3}, \ldots\}$ the family of totally disconnected graphs,

$\mathcal{S} = \{K_1, K_{1,1}, K_{1,2}, \ldots\}$ the family of stars, and
$\mathcal{P}= \{P_1, P_2, P_3, \ldots\}$ the family of paths.

All these families are increasing. For an arbitrary graph G, let $k(G, x)$, $i(G, x)$, $s(G, x)$, and $p(G, x)$ denote its *clique polynomial*, *independence polynomial*, *star polynomial*, and *path polynomial*, respectively. We wish to prove that all these polynomials are reconstructible from their polynomial decks. By Theorem 3 it suffices to show for each polynomial that for any graph of the corresponding increasing family of graphs its polynomial deck is unique.

Theorem 4 *The clique polynomial, the independence polynomial, the star polynomial, and the path polynomial are reconstructible from their polynomial decks.*

Proof. Let G be a graph on $n \geq 3$ vertices with $k(G_i, x) = k(K_{n-1}, x)$ for $i = 1, \ldots, n$. Note that $k(K_{n-1}, x)$ uniquely determines K_{n-1} because it tells that K_{n-1} is an induced subgraph of such a graph. Hence each G_i is isomorphic to K_{n-1}, which in turn implies that $G = K_n$.

Let G be a graph on $n \geq 3$ vertices with $i(G_i, x) = i(\overline{K_{n-1}}, x)$ for $i = 1, \ldots, n$. Since $i_1(G_i) = \binom{n-1}{2}$ it follows that the G_i are edgeless, but then G must be edgeless too.

Let G be a graph on $n \geq 3$ vertices such that $\{s(G_i, x), i = 1, \ldots, n\}$ coincides with the set of star polynomials of vertex-deleted subgraphs of $K_{1,n-1}$. Note that the number of edges is determined by these polynomials, and that it is $n - 1$. Since there is a G_i such that $s(G_i, x) = n - 1$, it follows that G_i is totally disconnected. Hence the remaining vertex of G must be incident with all $n - 1$ edges of G. This is only possible if G is $K_{1,n-1}$.

Finally, suppose that $p(G_i, x), i = 1, \ldots, n\}$ coincides with the set of path polynomials of the vertex-deleted subgraphs of P_n. As above we conclude that the number of edges of G is $n - 1$. Moreover, there are two subgraphs G_i, G_j with $p(G_i, x) = p(G_j, x) = p(P_{n-1}, x)$ which readily implies $G_i = G_j = P_{n-1}$. They have $n - 2$ edges, hence G must be obtained by adding to P_{n-1} one edge incident with a new vertex, thus G is a tree. All other subgraphs G_k are forests, and their polynomials $p(G_k, x)$ imply that they are disconnected with exactly two components (using that a forest has two components precisely when the number of vertices minus the number of edges equals 2). Hence $G = P_n$ as claimed. □

3 Reconstructing the cube polynomial and characterizing hypercubes

The n-cube Q_n, $n \geq 1$, is the graph with vertex set $\{0,1\}^n$, two vertices being adjacent if the corresponding tuples differ in precisely one place. We also set $Q_0 = K_1$. Let Q_n^- denote the graph obtained from Q_n by removing one of its vertices.

Let $\mathcal{Q}= \{Q_0, Q_1, Q_2, \ldots\}$ be the family of hypercubes. It is clear that $\mathcal{Q}$ is an increasing family of graphs. Following [3], let $\alpha_i(G)$ denotes the number of induced i-cubes of a graph G. Then the *cube polynomial* $c(G,x)$ of a graph G is

$$c(G,x) = \sum_{i\geq 0} \alpha_i(G)x^i \,.$$

By Remark 1 we obtain the following new characterization of hypercubes.

Proposition 5 *Let G be a graph. Then G is a hypercube if and only if for some $k \geq 0$, $c(G,x) = (x+2)^k$.*

Now the main result of this section.

Theorem 6 *The cube polynomial is reconstructible from its polynomial deck.*

Proof. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, and $G_i = G - v_i$, $1 \leq i \leq n$. By Theorem 3 it is sufficient to prove that every k-cube is uniquely determined by the deck of all $\mathcal{Q}$-polynomials of its vertex-deleted subgraphs.

Let G be a graph with the same polynomial deck (with respect to the cube polynomial) as Q_d, that is,

$$c(G_i,x) = c(Q_d^-,x) = \sum_{k=0}^{d} \binom{d}{k}(2^{d-k}-1)x^k \,,$$

for $i = 1, 2, \ldots, n$. Since $c(Q_d,x) = (x+2)^d$, we get

$$c(G_i,x) = \sum_{k=0}^{d} \binom{d}{k} 2^{d-k}x^k - \sum_{k=0}^{d} \binom{d}{k} x^k = c(Q_d,x) - \sum_{k=0}^{d} \binom{d}{k} x^k \,. \tag{3}$$

We claim that $G = Q_d$ and so $c(G,x) = (x+2)^d$. The claim is clear for $d = 2$, since then G has four vertices of degree two. Hence in the rest of the proof we assume $d \geq 3$.

Let $0 \leq k \leq d-1$. By Kelly's lemma we can deduce from the polynomial deck that G and Q_d have the same number of induced k-cubes. Hence (3) implies that after an arbitrary vertex of G is removed, the number of induced k-cubes is reduced by $\binom{d}{k}$. In other words, every vertex of G is contained in $\binom{d}{k}$ k-cubes. In particular, G is a d-regular graph.

Let u be an arbitrary vertex of G. Then, by the above, u is contained in a subgraph isomorphic to Q_{d-1} which we denote by H. Since G is d-regular, every vertex of H has exactly one neighbor not in H. Let K be the subgraph of G induced by vertices of G not in H. Note that $|V(K)| = |V(H)| = 2^{d-1}$.

Suppose two vertices of H have a common neighbor in K. (This assumption will eventually lead us to a contradiction.) Then there is a vertex v in K that does not have a neighbor in H. Using (3) again, v also lies in a subgraph isomorphic to Q_{d-1} which we denote by U. Note that all neighbors of v are in K. We claim that $U \subseteq K$.

Let x be an arbitrary vertex of U. We prove the claim by induction on $s = d_U(v,x)$. For $s = 1$ this is clear since v has no neighbor in H. Let now $s \geq 2$. Since U is a $(d-1)$-cube, x has at least two neighbors, say x_1 and x_2, in U at distance $s-1$ from v. By the induction assumption, x_1 and x_2 belong to K. Then x belongs to K as well, for otherwise it would have two distinct neighbors in K (this is not possible because then the degree of x would be at least $d+1$). This proves the claim.

Combining the facts that $|V(K)| = 2^{d-1}$ and $U \subset K$, we infer that $K = U$. This is again not possible, since then the degree of v in G would be less than d. Hence the assumption that two vertices of H have a common neighbor in K leads to a contradiction. Therefore, the edges between H and K form a matching, let it be denoted M.

We show next that M induces an isomorphism between H and K. Let xy be an arbitrary edge of H and let x' and y' be the neighbors of x and y in K. We wish to show that $x'y'$ is an edge of K. The vertex x lies in $\binom{d}{2}$ 4-cycles of G and inside H there are $\binom{d-1}{2}$ such 4-cycles. The remaining $d-1$ such 4-cycles must have a nonempty intersection with K. Since M is a matching, any such cycle must contain an edge of H. The degree of x in H is $d-1$, hence any edge xw must yield a 4-cycle, in particular $x'y'$ must be an edge of K. As G has $d2^{d-1}$ edges, there are no other edges in G except those in H together with M and those in K induced by M. Thus M induces an isomorphism. Since H is a $(d-1)$-cube, we conclude that G is a d-cube. $\square$

We continue with yet another characterization of hypercubes. (For other characterizations of hypercubes see [2,5,11,21,23].) For this purpose we invoke the following result of Mulder [22, page 55] about $(0,2)$-graphs; cf. also [19]. (A connected graph G is a $(0,2)$-*graph* if any two distinct vertices in G have exactly two common neighbors or none at all, cf. [20,22].)

Theorem 7 *Let G be a d-regular $(0,2)$-graph. Then $|V(G)| = 2^d$ if and only if $G = Q_d$.*

Corollary 8 *Let G be a $K_{2,3}$- and K_3-free graph on 2^d vertices with the largest degree d. Then G contains at most $2^{d-2}\binom{d}{2}$ 4-cycles. Equality holds if and only if $G = Q_d$.*

Proof. Let u be a vertex of G. Since G is K_3-free, any 4-cycle containing u also contains a vertex at distance 2 from u. Let $X(u)$ be the set of vertices v of G such that u and v lie in a common 4-cycle and $d(u,v) = 2$. Because G is $K_{2,3}$-free, any vertex of $X(u)$ determines a unique 4-cycle containing u. By the degree assumption there are at most $d(d-1)$ vertices at distance 2 from u, hence by the above u lies in at most $d(d-1)/2 = \binom{d}{2}$ 4-cycles. Consequently, G contains at most

$$\frac{|V(G)|\binom{d}{2}}{4} = 2^{d-2}\binom{d}{2}$$

4-cycles. Suppose that equality holds. Then every vertex is in exactly $\binom{d}{2}$ 4-cycles. This implies that G must be a d-regular $(0,2)$-graph. By Theorem 7 we infer that $G = Q_d$. □

4 Concluding remarks

We have considered five increasing families of graphs whose counting polynomials are reconstructible. These families are rather natural and we are sure other such families exist.

The reader might ask whether one can prove the reconstruction conjecture for some particular classes of graphs by using $\mathcal{H}$-polynomials that uniquely determine graphs of these classes. For trees with respect to the path and star polynomial the answer is negative.

The counterexample is not difficult to describe: Let P be a path of length four (i.e. on five vertices) with center p, and Q be obtained from $K_{1,3}$ by subdivision of an edge by a vertex q. We join p and q by an edge and add a pendant edge either to p or q to obtain the trees T_p and T_q, respectively; see Fig. 1. It is easy to see that T_p and T_q are not isomorphic, nevertheless they have the same path and star polynomials.

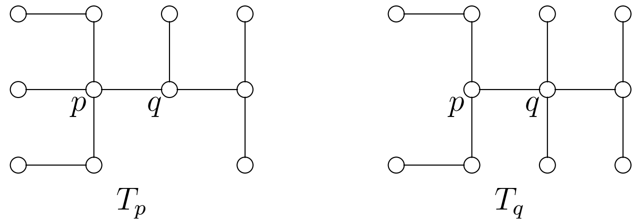

Fig. 1. Nonisomorphic trees

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References

[1] L. Babai, Automorphism Groups, isomorphism, reconstruction, (In: *Handbook of Combinatorics*, R.L. Graham. M. Grötschel, L. Lovász, eds.) Elsevier, Amsterdam, 1995, 1447–1540.

[2] A. Berrachedi and M. Mollard, Median graphs and hypercubes, some new characterizations, Discrete Math. 208/209 (1999) 71–75.

[3] B. Brešar, S. Klavžar, and R. Škrekovski, *The cube polynomial and its derivatives: the case of median graphs*, Electron. J. Combin. **10** (2003) #R3, 11pp.

[4] J.I. Brown and R.J. Nowakowski, Bounding the roots of independence polynomials, Ars Combin. 58 (2001) 113–120.

[5] M. Buratti, Edge-colourings characterizing a class of Cayley graphs and a new characterization of hypercubes, Discrete Math. 161 (1996) 291–295.

[6] E.J. Farrell, On a general class of graph polynomials, J. Combin. Theory Ser. B 26 (1979) 111–122.

[7] E.J. Farrell, On F-polynomials and reconstruction (In: *Advances in graph theory*, V.R. Kulli, ed.) Vishwa, Gulbarga, 1991, 155–162.

[8] E.J. Farrell, The impact of F-polynomials in graph theory (In: *Quo vadis, graph theory?*,) Ann. Discrete Math. 55, North-Holland, Amsterdam, 1993, 173–178.

[9] E.J. Farrell and S.A. Wahid, On the reconstruction of the matching polynomial and the reconstruction conjecture, Internat. J. Math. Math. Sci. 10 (1987) 155–162.

[10] D.C. Fisher and A.E. Solow, Dependence polynomials, Discrete Math. 82 (1990) 251–258.

[11] S. Foldes, A characterization of hypercubes, Discrete Math. 17 (1977) 155–159.

[12] C.D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.

[13] C.D. Godsil and I. Gutman, On the theory of the matching polynomial, J. Graph Theory 5 (1981) 285–297.

[14] I. Gutman, An identity for the independence polynomials of trees, Publ. Inst. Math. (Beograd) (N.S.) 50 (1991) 19–23.

[15] I. Gutman and D. Cvetković, The reconstruction problem for the characteristic polynomial of graphs, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Fiz. 498 (541) (1975) 45–48.

[16] E.M. Hagos, The characteristic polynomial of a graph is reconstructible from the characteristic polynomials of its vertex-deleted subgraphs and their complements, Electron. J. Combin. 7 (2000), #R12, 9 pp.

[17] C. Hoede and X. Li, Clique polynomials and independent set polynomials of graphs, Discrete Math. 125 (1994) 219–228.

[18] W.L. Kocay, Some new methods in reconstruction theory, (In: *Combinatorial mathematics, IX*, Brisbane, 1981), Lecture Notes in Math., 952, Springer, Berlin-New York, 1982, 89–114.

[19] M. Mollard, Two characterizations of generalized hypercube, Discrete Math. 93 (1991) 63–74.

[20] H.M. Mulder, $(0,\lambda)$-graphs and n-cubes, Discrete Math. 28 (1979) 179–188.

[21] H.M. Mulder, n-Cubes and median graphs, J. Graph Theory 4 (1980) 107–110.

[22] H.M. Mulder, *The Interval Function of a Graph*, Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.

[23] R. Scapellato, On F-geodetic graphs, Discrete Math. 80 (1990) 313–325.

[24] A.J. Schwenk, Spectral reconstruction problems, Ann. N.Y. Acad. Sci. 328 (1979) 183–189.

[25] S.K. Simić, A note on reconstructing the characteristic polynomial of a graph, (In: *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity*, Prachatice, 1990), Ann. Discrete Math., 51, North-Holland, Amsterdam, 1992, 315–319.

[26] W.T. Tutte, All the king's horses (in *Graph Theory and Related Topics*, J.A. Bondy, U.S.R. Murty, eds.) Academic Press, New York, 1979, 15–33.