Distinguishing Cartesian powers of graphs

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Abstract

The distinguishing number $D(G)$ of a graph is the least integer d such that there is a d-labeling of the vertices of G that is not preserved by any nontrivial automorphism of G. We show that the distinguishing number of the square and higher powers of a connected graph $G \neq K_2, K_3$ with respect to the Cartesian product is 2. This result strengthens results of Albertson [1] on powers of prime graphs, and results of Klavžar and Zhu [14]. More generally, we also prove that $d(G \Box H) = 2$ if G and H are relatively prime and $|H| \leq |G| < 2^{|H|} - |H|$. Under additional conditions similar results hold for powers of graphs with respect to the strong and the direct product.

Key words: Distinguishing number; Graph automorphism; Products of graphs

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1 Introduction

A labeling $\ell : V(G) \rightarrow \{1, 2, \ldots, d\}$ of a graph G is *d-distinguishing* if no nontrivial automorphism of G preserves the labeling. The *distinguishing number* $D(G)$ of a graph G is the least integer d such that G has a d-distinguishing labeling. This concept was introduced by Albertson and Collins in [2] and has received considerable attention, cf. [4, 5, 7, 16, 18].

All graphs in this paper are assumed to be connected. This is possible without loss of generality because a graph and its complement have the same automorphism groups (and hence equal distinguishing numbers) and because the complement of a disconnected graph is connected.

If a graph has no nontrivial automorphisms its distinguishing number is 1. In other words, $D(G) = 1$ for asymmetric graphs. The other extreme, $D(G) = |G|$, occurs if and only if $G = K_n$. This follows from the fact that $D(G) \leq \Delta(G)$ for all graphs $G \neq K_n$, $K_{n,n}$ and C_5 (see [13]).

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The Cartesian product (and other products) of graphs have automorphism groups that are well understood. Hence it is not surprising that the distinguishing number of Cartesian product graphs have been thoroughly investigated.

It all started with the paper [3] of Bogstad and Cowen in which the distinguishing number of hypercubes was determined: $D(Q_2) = D(Q_3) = 3$ and $D(Q_d) = 2$ for $d \geq 4$. Now, hypercubes are the simplest instances of Cartesian product graphs, that is, $Q_d = K_2^d$, where G^r stands for the rth power of G with respect to the Cartesian product. Then Albertson [1] proved that for a connected prime graph G, $D(G^r) = 2$ for all $r \geq 4$, and, if $|V(G)| \geq 5$, then $D(G^r) = 2$ for all $r \geq 3$. This considerably generalizes the results of [3]. Moreover, Albertson conjectured that for any connected graph G there exists an integer $R = R(G)$ such that for any $r \geq R$, $D(G^r) = 2$. This conjecture has been then verified in [14], where it was shown that $D(G^r) = 2$ for any connected graph $G \neq K_2$ and any $r \geq 3$. Here we round out these investigations with the following theorem that includes second powers as well.

Theorem 1.1 *Let $G \neq K_2, K_3$ be a connected graph and $k \geq 2$. Then $D(G^k) = 2$.*

As we already mentioned, the case $G = K_2$ has been settled in [3], whereas $D(K_3^q) = 2$ for $q \geq 3$ by [14]. It is also known (and can be checked directly) that $D(K_3^2) = 3$. Hence Theorem 1.1 completely determines the distinguishing number of all Cartesian powers—it is always two, with the exception of the three special cases K_2^2, K_2^3 and K_3^2, whose distinguishing number is three.

Our proof of Theorem 1.1 does not use the motion lemma of [16] (or its modification from [14]), and is self-contained in the sense that it mainly relies on the properties of the automorphism groups of Cartesian products of prime and relatively prime graphs. We will describe these results in Section 2 (see [12] for details). Then, in Section 3, we prove Theorem 1.1. In Section 4 we consider Cartesian products with factors of different sizes and prove that $D(G \square H) = 2$ if G and H are relatively prime and $|H| \leq |G| < 2^{|H|} - |H|$. In the last section we show that similar results hold for powers of graphs with respect to the strong and the direct product.

In the sequel we will label graphs with two or more labels and will, in the case of two labels, sometimes utilize the binary representation of numbers. We shall also consider 2-labelings as mappings of the form $\ell : V(G) \to \{0, 1\}$. Alternatively, we will speak of black and white colors or regard a labeling or coloring as a partition of $V(G)$.

2 Algebraic properties of Cartesian products

Let us recall that the *Cartesian product* $G \square H$ of two graphs has the vertex set $V(G) \times V(H)$ where the vertex (g, h) is adjacent to (g', h') whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. If $(g, h) \in G \square H$ we set $p_G(g, h) = g$ and $p_H(g, h) = h$. The mappings $p_G : V(G \square H) \to V(G)$ and $p_H : V(G \square H) \to V(H)$ are called *projections* of $G \square H$ onto the respective factors.

A graph is called *prime* (with respect to the Cartesian product) if it cannot be represented as the Cartesian product of two nontrivial graphs. Clearly every graph is a product of prime graphs. It is well known that this prime factor decomposition is unique for connected graphs [17, 19], see also [8, 12]. That is, every connected graph G can be uniquely represented as a product of prime graphs G_i

$$G = G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_k,$$

up to the order and isomorphisms of the factors.

Two graphs G and H are *relatively prime* (with respect to the Cartesian product) if there is no nontrivial graph that is a factor of both G and H. Clearly, two prime graphs are relatively prime.

If $G = G_1 \,\square\, G_2$ and $\alpha \in \mathrm{Aut}(G_1)$, then the mapping

$$\alpha^* : V(G_1 \,\square\, G_2) \to V(G_1 \,\square\, G_2)$$

defined by

$$\alpha^* : (g_1, g_2) \mapsto (\alpha g_1, g_2)$$

is an automorphism of G.

Furthermore, if $G_1 = G_2$, that is, if $G = G_1 \,\square\, G_1$, then

$$\beta : (g_1, g_2) \mapsto (g_2, g_1)$$

also is an automorphism of G.

The automorphism α^* of G is induced by an automorphism of a factor and β by an interchange of isomorphic factors. By [10] such automorphisms generate $\mathrm{Aut}(G)$. One can thus visualize $\mathrm{Aut}(G)$ as the automorphism group of the disjoint union of the G_i.

A *fiber* $G_i^{(g_1,\ldots,g_k)}$ of $G_1 \,\square\, \cdots \,\square\, G_k$ is the subgraph induced by the vertex set

$$\{(g_1, g_2, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_k) \mid x \in G_i\}\,.$$

This set consists of all vertices of G that differ from $v = (g_1, \ldots, g_k)$ in the i-th coordinate. Clearly G_i^v is isomorphic to G_i and the number of G_i-fibers is equal to the number of vertices in

$$G_1 \,\square\, G_2 \,\square\, \cdots G_{i-1} \,\square\, G_{i+1} \,\square\, \cdots \,\square\, G_k\,.$$

Any two G_i-fibers are either identical or disjoint.

Every nontrivial automorphism α^* of G that is induced by an automorphism α of G_i preserves every single G_i-fiber and permutes the set of G_j-fibers for every $j \neq i$. Furthermore, any automorphism β of G that is induced by an interchange of the (isomorphic) factors G_i and G_j interchanges the set of G_i-fibers with the set of G_j-fibers. It also stabilizes the sets of G_r-fibers for any factor G_r with $r \neq i, j$.

In particular this implies that in a product $G \,\square\, H$ of relatively prime graphs every automorphism preserves the set of G-fibers and the set of H-fibers, cf. [12, Corollary 4.17].

As an example consider $K_2 \,\square\, K_3$ (see Fig. 1). The K_3-fibers have the vertex sets $\{a, b, c\}$, $\{a', b', c'\}$, and the K_2-fibers the vertex sets $\{a, a'\}$, $\{b, b'\}$ and $\{c, c'\}$. Since

K_2 and K_3 are relatively prime, every automorphism of $K_2 \Box K_3$ either stabilizes both sets $\{a, b, c\}$ and $\{a', b', c'\}$ or interchanges them. Similarly it permutes (or stabilizes) the sets $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$. If we color a, b and b' white and the other vertices black, cf. Fig.1, then the sets

$$\{a, b, c\}, \ \{a', b', c'\}$$

cannot be interchanged because they have different numbers of black and white vertices. The same holds for the sets $\{a, a'\}$, $\{b, b'\}$ and $\{c, c'\}$. Thus $D(K_2 \Box K_3) = 2$.

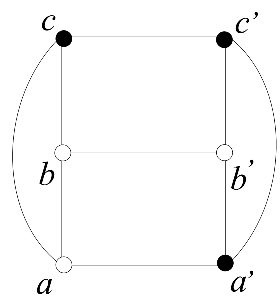

Figure 1: A 2-distinguishing labeling of $K_2 \Box K_3$

The same argument shows that $D(K_2 \Box P_3) = 2$.

3 Proof of the main theorem

We begin with powers of prime graphs.

Lemma 3.1 *Let G be a connected prime graph on at least four vertices. Then $D(G \Box G) = 2$.*

Proof. Let $4 \leq k = |G|$, $H \cong G$, and $V(G) = V(H) = \{1, \ldots, k\}$. Color the vertices $(k-1, k-1)$, (k, k), the vertices (i, j) with $1 \leq i < j \leq k$, and vertices $(i, i-2)$ for $3 \leq i \leq k$ black and the other ones white.

Since G is prime, all automorphisms of $G \Box H$ are generated by automorphism of G or H or interchanges of the G-fibers with the H-fibers.

Automorphisms of the product generated by automorphism of G preserve the number of black vertices in every G-fiber and the automorphisms of $G \Box H$ generated by automorphisms of H permute them. Thus, such automorphisms preserve or permute the number of black vertices in every G-fiber. Similarly one shows that these automorphisms preserve or permute the number of black vertices in the H-fibers.

As there is one G-fiber all of whose vertices are black but no H-fiber with this property, we infer that our coloring forbids interchange of the G-fibers with the H-fibers.

Moreover, any two G-fibers have different numbers of black vertices, and thus have to be stabilized by every color preserving automorphism of $G \Box H$. If $k = 4, 5$ we easily check directly that H-fibers cannot be interchanged. Let $k \geq 6$, then there are two H-fibers with $k-2$ black vertices and two with just three, but any other two H-fibers have different numbers of black vertices. These pairs of H-fibers are $H^{(2,1)}$, $H^{(3,1)}$ and $H^{(k-2,1)}$, $H^{(k-1,1)}$. It is easy to see that they cannot be interchanged since $(2,1)$ is white, but $(3,1)$ is black, and because $(k-2,2)$ is black but $(k-1,2)$ is white. $\square$

This implies, in particular, that $D(K_k \,\square\, K_k) = 2$ for $k \geq 4$. On the other hand it is not hard to show that $D(K_3 \,\square\, K_3) = 3$.

Lemma 3.2 *Let G, H be connected graphs with $3 \leq |G| \leq |H| + 1$. If G is prime and $D(H) \geq 2$, then $D(G \,\square\, H) \leq D(H)$.*

Proof. Since $D(H) \geq 2$ we have at least the colors black and white at our disposal. Color one H-fiber completely black, one completely white and endow one with the distinguishing coloring. (This is possible since there are at least three H-fibers.) If there are more H-fibers color them such that any two H-fibers, including the ones we have already colored, have different numbers of black vertices. Since $|G| \leq |H| + 1$ this is possible.

If H is not prime, it may have a prime factor H' isomorphic to G, in this case $G \,\square\, H$ has an automorphism interchanging G with H'. But then every G-fiber is mapped into an H'-fiber. Thus also every H'-fiber completely contained in the H-fiber that is completely black must be an image of a G-fiber. Since there is no completely black G-fiber this is not possible. Therefore our coloring requires that the G-fibers are mapped into G-fibers and hence all H-fibers into H-fibers. These fibers have pairwise different numbers of black vertices and must thus be stabilized. This includes the fiber with the distinguishing coloring. But then all G-fibers have to be stabilized and, moreover, fixed because the H-fibers are stabilized. $\square$

Corollary 3.3 *Let G be a connected prime graph on at least 4 vertices. Then $D(G^k) = 2$ for $k \geq 2$.*

Proof. For $k = 2$ this is Lemma 3.1, for $k > 2$ use induction and replace H by G^{k-1} in Lemma 3.2. $\square$

We continue with a short proof of the following lemma from [14].

Lemma 3.4 *Let G and H be connected, relatively prime graphs with $D(G) = 2$ and $2 \leq D(H) \leq 3$. Then $D(G \,\square\, H) = 2$.*

Proof. Let G, H be relatively prime connected graphs, $\ell_G = (A_1, A_2)$ a distinguishing 2-labeling of G and $\ell_H = (B_1, B_2, B_3)$ a distinguishing 3-labeling of H. We define a 2-labeling ℓ of $G \,\square\, H$ by coloring the vertices from

$$(A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_2)$$

white and all the other vertices black.

With this coloring all vertices of fibers G^u with $p_H(u) \in B_3$ are black, those with $p_H(u) \in B_2$ have only white vertices and the other G-fibers have both black and white ones. Clearly they form blocks, so every automorphism of $G \square H$ induced by one of H must preserve the blocks B_1, B_2, B_3 of H, and is thus the identity.

The H^u-fibers are divided into two classes, those with $p_G(u) \in A_1$ and the ones with $p_G(u) \in A_2$. The former have $|B_1| + |B_2|$ white vertices, the latter only $|B_2|$. These classes are stabilized by any automorphism of $G \square H$. By the same argument as before every automorphism of $G \square H$ that respects the 2-labeling and is induced by an automorphism of H must be the identity.

Since $\mathrm{Aut}(G \square H)$ is generated by $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, ℓ is a distinguishing 2-labeling.

The observation that the above reasoning also holds if B_3 is empty completes the proof. □

Note that the labeling from Fig. 1 is a special case of the construction in the above proof.

Lemma 3.5 *Let G be a connected graph. If G has a prime factor of cardinality at least 4, and no factor K_2, then $D(G^k) = 2$ for $k \geq 2$.*

Proof. Let $G = G_1^{p_1} \square \cdots \square G_r^{p_r}$ be the prime factor decomposition of G, where the G_i's are prime graphs and $p_i \geq 1$. Suppose first that all G_i have at least four vertices. Then $D(G_i^{kp_i}) = 2$ by Corollary 3.3. Since $G^k = G_1^{kp_1} \square \cdots \square G_r^{kp_r}$ successive applications of Lemma 3.4 show that $D(G^k) = 2$ in this case.

If we also have a factor $G_0^{p_0}$, where $|G_0| = 3$, we complete the proof by successive applications of Lemma 3.2. □

To be able to treat the case where G contains a factor K_2^p we invoke a slightly different version of Lemma 3.2.

Lemma 3.6 *Let G, H be connected graphs with $3 \leq |G| \leq |H| + 1$. If G and H are relatively prime, then $D(G \square H) \leq \max\{2, D(H)\}$.*

Proof. By the same arguments as in the proof of Lemma 3.2. □

Lemma 3.7 *Let $X = K_2^p \square Y$, where Y is a connected graph relatively prime to K_2 and $D(Y^k) = 2$ for $k \geq 2$. Then $D(X^k) = 2$.*

Proof. We consider the case where $p = 1$ first. Set $G = K_2^k$ and $H = Y^k$. Note that G and H are relatively prime. Since $|G| = 2^k < |Y|^k$ we can apply Lemma 3.6.

For $p > 1$ we note that $D(K_2^r) = 2$ for $r \geq 4$, as has been shown in [3]. Now an application of Lemma 3.4 completes the proof. □

We now complete the proof of the main theorem. In view of Lemmas 3.5 and 3.7 it remains to consider graphs G whose prime factors have two or three vertices. It is easily seen that $D(P_3 \Box P_3) = 2$. By Lemma 3.2 all higher powers of P_3 have distinguishing number 2. Therefore, if G has a prime factor P_3, Lemma 3.4 implies that $D(G^k) = 2$ for $k \geq 2$.

We have already mentioned that $D(K_3^2) = 3$. A 2-distinguishing labeling of K_3^3 can be constructed as follows. Let G_1, G_2, G_3 be three copies of K_3. Choose a distinguishing labelling of $H = G_2 \Box G_3$ with three colors, say b, w, and g, and let b, w, and g also denote the number of vertices colored with the respective colors. Let $g \leq w \leq b$, then $g + b > w$. Color two H-fibers of $G_1 \Box H$ with the distinguishing coloring. In one fiber, change g to b. In the other, interchange b and w and change g to w. Color the remaining H-fiber of $G_1 \Box H$ completely with w. It is readily verified that the constructed labeling is distinguishing.

Since $D(K_3^3) = 2$, applications of Lemma 3.2 yield $D(K_3^k) = 2$ for any $k \geq 3$. As in addition $D(K_2^k) = 2$ for any $k \geq 4$, $D(G^k) = 2$ for any $k \geq 2$ and any graph G that has a prime factor K_3^r of K_2^r for some $r \geq 2$. The only remaining case is $G = K_2 \Box K_3$. It is straightforward to see that $D(K_2^2 \Box K_3) = 2$. Then apply Lemma 3.2 with $G = K_3$ to infer that $D(G^2) = 2$. Finally, $D(G^k) = 2$ for $k > 2$ by Lemma 3.6. □

4 Products with factors of different sizes

In this section we show that the distinguishing number of the product of two relatively prime graphs is 2 if their sizes do not differ too much.

Lemma 4.1 *Let $k \geq 2$, let G be a connected graph on $2^k - k + 1$ vertices and H a connected graph on k vertices that is relatively prime to G. Then $D(G \Box H) \leq 2$.*

Proof. Since G and H are relatively prime every automorphism maps G-fibers into G-fibers and H-fibers into H-fibers.

We wish to color the G-fibers with two colors such that the number of ones in the fibers is $2^{k-1}, 2^{k-1} - 1, \ldots, 2^{k-1} - k + 1$. To this end we consider all binary numbers from 0 to $2^k - 1$ and remove the numbers $2 - 1, 2^2 - 1, \ldots, 2^{k-1} - 1$. Let $\mathcal{B}_k$ denote the set of these numbers. Clearly, $|\mathcal{B}_k| = 2^k - k + 1$.

Regard the binary numbers from $\mathcal{B}_k$ as vectors of length k and label the H-fibers with them. Then the number of ones in the G-fibers is $2^{k-1}, 2^{k-1} - 1, \ldots, 2^{k-1} - k + 1$. They are different, thus any automorphism α of $G \Box H$ that respects the 0-1 labeling must preserve G-fibers, so α can only interchange H-fibers (as 0-1 vectors). They are all different, hence α is the identity. □

Let $b = b_1 b_2 \ldots b_n$ and $c = c_1 c_2 \ldots c_n$ be binary numbers. Then we say that c is the *binary complement* of b if $b_i + c_i = 1$, $1 \leq i \leq n$.

Theorem 4.2 *Let $k \geq 2$ and let G and H be connected, relatively prime graphs with $k \leq |H| \leq |G| \leq 2^k - k + 1$. Then $D(G \Box H) \leq 2$.*

Proof. Suppose first that $|H| = k$. Let $\mathcal{B}_k$ be the set of $2^k - k + 1$ binary numbers as in the proof of Lemma 4.1. Note that there are $(2^k - 2(k-1))/2 = 2^{k-1} - (k-1)$ pairs of binary complements in $\mathcal{B}_k$, one of them being the pair $\{00\ldots0, 11\ldots1\}$. Set $x = (2^k - (k-1) - |G|)/2$. If x is an integer, then let $\mathcal{C}_k$ be the set of binary numbers obtained from $\mathcal{B}_k$ by removing $2x$ numbers that form x complementary pairs. If x is not an integer, remove $00\ldots0$, and then $\lfloor x \rfloor$ complementary pairs. Note that $|\mathcal{C}_k| = |G|$ and consider the binary numbers from $\mathcal{C}_k$ as vectors of length k and label the H-fibers with them. Recall that the vectors from $\mathcal{B}_k$ have pairwise different number of ones in each of their k coordinates. Since in the construction of $\mathcal{C}_k$ we have removed binary complements (and possibly also $00\ldots0$), the vectors from $\mathcal{C}_k$ also have pairwise different numbers of ones in their coordinates. Thus any automorphism α of $G \,\Box\, H$ must preserve G-fibers and since the H-fibers are pairwise different, α is the identity.

Suppose next that $k < |H|$ (and $|H| \leq |G|$). Then select a subgraph H' of H with k vertices and use the above construction for $G \,\Box\, H'$. This construction leads to k different numbers of 1s in fibers $G^{(g,h')}$, where $h' \in H'$. Let S be the set of these numbers. Now label the G-fibers of $G \,\Box\, (H \setminus H')$ arbitrarily with 0s and 1s such that every fiber has a distinct number of 1s from the set $\{0, 1, \ldots, 2^k - k + 1\} \setminus S$. As before the G-fibers and the H-fibers are fixed by every automorphism. □

Note that Theorem 4.2 holds for $k = 2$ by default, because then $G = H = K_2$, and G, H are not relatively prime. In fact, we already mentioned that $D(K_2 \Box K_2) = 3$.

In contrast to Theorem 4.2 we have the following result.

Theorem 4.3 *$D(K_m \,\Box\, K_n) \geq 3$ for $m \geq 2$ and $n > 2^m$.*

Proof. Let ℓ be an arbitrary 2-coloring of $K_m \,\Box\, K_n$. Since there are more than 2^m K_m-fibers, at least two of them have identical 2-colorings, that is, if K_m^u and K_m^v are these two fibers and $x \in K_m^u$, $y \in K_m^v$ have the same projections onto K_m, then $\ell(x) = \ell(y)$. Since $\mathrm{Aut}(K_m \,\Box\, K_n)$ acts transitively on the K_m-fibers we infer that ℓ is not distinguishing. □

Recall that $D(K_2 \,\Box\, K_4) = 3$. In addition, we can show that $D(K_3 \,\Box\, K_7) = 3$. Hence it seems to be an interesting question whether there are cases where $D(K_m \,\Box\, K_n) = 2$ for $2^m - m + 1 < n \leq 2^m$.

5 Distinguishing strong and direct products

The results for the Cartesian product depend on the structure of the automorphism group of the product, in some cases on the size of the factors, and, of course, on the unique prime factorization property. Let us check when these conditions are satisfied for the strong and the direct product.

We begin with the definition of these products. Both the strong product $G \boxtimes H$ and the direct product $G \times H$ of two graphs have the same vertex sets as the Cartesian product. In the case of the direct product two vertices $u, v \in G \times H$ are joined by an edge if $[p_G(u), p_G(v)] \in E(G)$ and $[p_H(u), p_H(v)] \in E(H)$, whereas the edge set of the strong product is the union of the edge sets of the Cartesian and the direct product. Both products are commutative and associative. Moreover, every connected graph has a unique prime factor decomposition with respect to the strong product [6], cf. also [9, 11, 12].

5.1 Distinguishing strong products

The structure of the automorphism group of strong products is generally not the same as that of Cartesian products. The most striking example are strong products of complete graphs. We have

$$K_m \boxtimes K_n = K_{m \cdot n}$$

and thus $D(K_n \boxtimes K_n) = n^2$, whereas $D(K_n \Box K_n) = 2$ for any $n \geq 4$. The reason is that any two vertices u, v of K_n are adjacent and similar, that is, any vertex $w \neq u, v$ is either adjacent to both u and v or to neither one of them. A graph is called *S-thin* if it has no pairs of such vertices.

The prime factors of S-thin graphs are thin again and the structure of the automorphism group of strong products of connected, prime, S-thin graphs is the same as that od corresponding Cartesian products. Since K_2 and K_3 are not thin, we have the following theorem.

Theorem 5.1 *Let G be a connected, S-thin graph and $\boxtimes G^k$ the k-th power of G with respect to the strong product. Then $D(\boxtimes G^k) = 2$ for $k \geq 2$.*

5.2 Distinguishing direct products

Connected nonbipartite graphs have unique prime factor decompositions with respect to the direct product [15], see also [11]. If such a graph G has no pairs u, v of vertices with the same closed neighborhoods, then the structure of the automorphism group of G depends on that of its prime factors exactly as in the case of the Cartesian product.

Graphs with no pairs of vertices with the same closed neighborhoods are called *R-thin.*

Theorem 5.2 *Let G be a nonbipartite, connected, R-thin graph different from K_3 and $\times G^k$ the k-th power of G with respect to the direct product. Then $D(\times G^k) = 2$ for $k \geq 2$.*

For the case $G = K_3$ we use the fact that $\times K_3^k$ and K_3^k have the same automorphism groups. Hence $D(K_3 \times K_3) = 3$ and $D(\times K_3^k) = 2$ for $k \geq 3$.

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