Fast recognition of classes of almost-median graphs

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Abstract

In this paper it is shown that a class of almost-median graphs that includes all planar almost-median graphs can be recognized in $O(m \log n)$ time, where n denotes the number of vertices and m the number of edges. In particular, planar almost-median graphs can be recognized in linear time. As a key auxiliary result we prove that all bipartite outerplanar graphs are isometric subgraphs of the hypercube and that the embedding can be effected in linear time.

1 Introduction and preliminaries

This paper is concerned with the recognition of classes of isometric subgraphs of hypercubes. Such graphs are of considerable interest in diverse applications and constitute a rich class of graphs. Some of them, for example planar median graphs, can be recognized in linear time, others can be recognized within the same time complexity as triangle-free graphs, but several important simply definable classes, such as planar isometric subgraphs of hypercubes, can only be recognized in $O(mn)$ time, where m denotes the number of edges and n the number of vertices of the graph under consideration.

As a step towards the improvement of the complexity of recognizing planar isometric subgraphs of hypercubes we present a class of almost-median graphs that can be recognized in $O(m \log n)$ time. In particular we show that all planar almost-median graphs can be recognized in linear time.

The methods for the solution of this problem extend those of a paper of Brešar, Imrich, and Klavžar [4] and make use and prove the fact that bipartite outerplanar graphs are isometric subgraphs of hypercubes and that they can be embedded in linear time. Outerplanar graphs will be treated in Section 2, the main result follows in Section 3.

We continue with definitions of several basic graph theoretical concepts and refer to standard texts or to [13] for the terms not listed here.

The *distance* $d_G(u,v)$, or briefly $d(u,v)$, between two vertices u and v in a graph G is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric*, if $d_H(u,v) = d_G(u,v)$ for all $u,v \in V(H)$. It is *convex* if for every $u,v \in V(H)$ all shortest u,v-paths belong to H. Convex subgraphs are isometric.

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a,x) is adjacent to (b,y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$.

The products $K_2 \Box C_n$, $n \geq 3$, are called *prisms* and the products $K_2 \Box P_n$ *ladders*. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes*.

An important subclass of partial cubes are median graphs. A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w of G such that x lies on a shortest u,v-path, a shortest u,w-path, and on a shortest v,w-path.

For partial cubes, the sets U_{ab} and F_{ab} that we shall define below play a crucial role. Let ab be an edge of connected, bipartite graph $G = (V,E)$. Then

$$W_{ab} = \{w \in V \mid d_G(a,w) < d_G(b,w)\},$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\},$$
$$F_{ab} = \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.$$

We will denote the subgraphs induced by the vertices of U_{ab} by $\langle U_{ab} \rangle$.

It follows from results in [1] that median graphs are precisely the bipartite graphs for that all $\langle U_{ab} \rangle$ are convex. By this result, the following definitions from [12] make sense.

A bipartite graph is a *semi-median* graph if it is a partial cube in that all $\langle U_{ab} \rangle$ are connected.

Similarly, a bipartite graph is *almost-median* if it is a partial cube for that every $\langle U_{ab} \rangle$ is an isometric subgraph of G. It is clear that median graphs are almost-median graphs, that almost-median graphs are semi-median graphs, and that semi-median graphs are partial cubes.

One of the most useful relations for the investigation of metric properties of graphs in general, partial cubes, and Cartesian products in particular is the Djoković-Winkler relation Θ, (cf. [8, 17]). Two edges $e = xy$ and $f = uv$ of G are in the relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

Clearly, Θ is reflexive and symmetric. Its transitive closure will be denoted by Θ^*. In general $\Theta \neq \Theta^*$.

We continue with several basic and well-known properties of Θ that we shall use in the sequel, (cf. [13]; for Lemma 4 see [9]).

Lemma 1 *Suppose that a walk P connects the endpoints of an edge e but does not contain it. Then P contains an edge f with $e\Theta f$.*

Lemma 2 *Let P be a shortest path. Then no two edges of P are in relation Θ.*

Lemma 3 *Let G be a bipartite graph and $e = uv$, $f = xy$ be two edges of G with $e\Theta f$. Then the notation can be chosen such that $d(u, x) = d(v, y) = d(u, y) - 1 = d(v, x) - 1$.*

Lemma 4 *Let e and f be edges from different blocks of a graph G. Then e is not in relation Θ with f.*

The following characterization of partial cubes is due to Winkler [17].

Theorem 5 *A bipartite graph is a partial cube if and only if Θ is transitive.*

Another relevant relation defined on the edge set of a graph is δ. We say an edge e is in relation δ to an edge f if $e = f$ or if e and f are opposite edges of a 4-cycle without diagonals.

Clearly δ is reflexive and symmetric. Moreover, it is contained in Θ. Thus its transitive closure δ^* is contained in Θ^*. In [12] it is shown that a bipartite graph is semi-median if and only if $\Theta = \delta^*$.

Suppose that $e = e_1 \delta e_2 \delta \dots \delta e_k = f$ is a sequence of edges by virtue of which e and f are in relation δ^*. If all endvertices of the e_i are different, the union of squares that contain e_i and e_{i+1}, where $i = 1, 2, \dots, k-1$, forms a ladder, that is, the Cartesian product of path of length $k-1$ by an edge. In such a case we shall frequently say that

e and f are connected by a "ladder". Clearly a ladder does not necessarily provide a shortest path between e and f.

Let $e = ab$ be an edge in a partial cube G. By F^{δ}_{ab} we denote the set of edges in G that are connected by some ladder with e. It is clear that $F^{\delta}_{ab} \subseteq F_{ab} = F^{\Theta}_{ab}$ and that $F^{\delta}_{ab} = F_{ab} = F^{\Theta}_{ab}$ for semi-median graphs. We will denote the vertices on the side of a in these ladders by U^{δ}_{ab} and by U^{δ}_{ba} the vertices of the side of b. Again $U^{\delta}_{ab} \subseteq U_{ab} = U^{\Theta}_{ab}$ in general and $U^{\delta}_{ab} = U_{ab} = U^{\Theta}_{ab}$ for semi-median graphs. By abuse of language we will use the notation U^{δ}_{ab} also for the graph that is obtained from the vertices of U^{δ}_{ab} and edges on one side of above mentioned ladders. Note that U^{δ}_{ab} is not necessarily the same as $\langle U_{ab} \rangle$. This occurs when $\left\langle U^{\delta}_{ab} \right\rangle$ and $\left\langle U^{\delta}_{ba} \right\rangle$ are not isomorphic, see Figure 1.

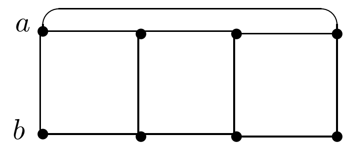

Figure 1: $\langle U_{ab} \rangle = C_4$ and $U^{\delta}_{ab} = P_4$.

2 Outerplanar graphs

In this section we define outerplanar graphs and present several of their characterizations. The main result of this section is Theorem 10 which asserts that connected bipartite outerplanar graphs are partial cubes.

Moreover, we will show how to embed outerplanar graphs into hypercubes in linear time.

A graph G is *planar* if it can be drawn in the plane such that any two edges have at most endpoints in common and the vertices are distinct points. Such drawings are called a *plane drawings* of G. Any plane drawing of G divides the plane into regions which are called *faces*. One of those faces is unbounded and is called the *exterior* or *outer* face, the others are *interior* faces.

A graph G is *outerplanar* if it is planar and embeddable into the plane such that all vertices lie on the outer face of the embedding. Such an embedding is called an *outerplanar embedding* of G.

To find the outerplanar embedding of an outerplanar graph we will use the following construction, cf. [15]. Let G^+ be the graph obtained from G by adding a new auxiliary vertex that is adjacent to all vertices of G. For a bipartite graph G the auxiliary vertex is the only vertex for which every incident edge is in a triangle.

If the outerplanar graph G is 1-connected the outer face is not unique. On the other hand, if G is 2-connected, then G^+ is 3-connected. It is well known that 3-connected graphs are uniquely embeddable into the plane.

The outer face of G is uniquely determined in the sense that its edges are uniquely determined. One observes that G is outerplanar if and only if G^+ is planar, cf. [15]. Indeed, if we remove the auxiliary vertex in G^+, the adjacent vertices form an outer cycle. For, if G is not outerplanar there exists a vertex in an inner face. Since this vertex must be adjacent to the auxiliary vertex, G^+ cannot be planar.

Let G be an outerplanar graph. Then we can embed G^+ in linear time into the plane by the algorithm of Hopcroft an Tarjan [10] and obtain an outerplanar embedding of G within the same time complexity by removal of the auxiliary vertex.

Proposition 6 *Outerplanar graphs can be recognized in linear time. Moreover, all faces of an outerplanar embedding and the outer face can be determined within the same time complexity.* □

We wish to mention that there are simple direct algorithms for testing whether a given graph is outerplanar [3, 16]. By the above construction it is not difficult to see that the two Kuratowski graphs are the only obstructions to outerplanarity. In [15] the following special case of Kuratowski's theorem can be found.

Theorem 7 *A graph is outerplanar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_4 or $K_{2,3}$.* □

In [2] it is shown that $G \Box K_2$ is planar if G is outerplanar. Moreover $G \Box K_2$ is planar if and only if G does not contain a subgraph that is isomorphic to the subdivision of K_4 or $K_{2,3}$. Thus we infer the following result from Theorem 7.

Proposition 8 *G is outerplanar if and only if $G \Box K_2$ is planar.* □

We continue with the following lemma.

Lemma 9 *Let G be a bipartite outerplanar graph and $e = u_0v_0$ and $f = u_kv_k$ be two edges in relation Θ, where $d(u_0, u_k) = d(v_0, v_k) = k$ and $d(u_0, v_k) = d(v_0, u_k) = k + 1$. Then the shortest paths from u_0 to u_k and v_0 to v_k are unique. Let them be $P = u_0u_1 \dots u_k$ and $Q = v_0v_1 \dots v_k$, respectively. Moreover every edge g with one endvertex on P and the other on Q is of the form $g = u_iv_i$ and thus in relation Θ with both e and f.*

Proof Let $e = u_0v_0$ and $f = u_kv_k$ be in relation Θ with $d(u_0, u_k) = d(v_0, v_k) = k$ and $d(u_0, v_k) = d(v_0, u_k) = k + 1$. Suppose $P = u_0u_1 \dots u_k$ is a shortest u_0, u_k-path and $Q = v_0v_1 \dots v_k$ a shortest v_0, v_k-path. We first show that P and Q are distinct. If $u_i = v_i$, then G is not bipartite, and if $u_i = v_j$, $i \neq j$, then $d(u_0, v_k) < k + 1$ for $i < j$ or $d(v_0, u_k) < k + 1$ for $i > j$, which is impossible.

If P or Q are not unique, G contains a subdivision of $K_{2,3}$ and cannot be outerplanar.

Now let $u_i v_j$ be an edge from P to Q. If $i = j$ we are done. If $i \neq j$, then $d(u_0, v_k) < k + 1$ for $i < j$ or $d(v_0, u_k) < k + 1$ for $i > j$. □

Theorem 10 *Connected bipartite outerplanar graphs are partial cubes.*

Proof We will show that Θ is transitive on G.

Suppose $e\Theta f$ and $f\Theta g$, where $e = uv$, $f = xy$, and $g = wz$ with $d(u,x) = d(v,y) = k$, $d(u,y) = d(v,x) = k+1$, $d(x,w) = d(y,z) = \ell$ and $d(x,z) = d(y,w) = \ell + 1$. Let P, Q, R, and S denote the shortest u,x-path, v,y-path, x,w-path, and y,z-path, respectively.

Suppose $w \in P$. If $z \in P$, then g is an edge of P and cannot be in relation Θ with f. Since G is outerplanar, the vertex z must lie on Q and $e\Theta g$ by Lemma 9.

If w is on Q, then $d(y,w) = \ell + 1$ and $d(v,w) = k - \ell - 1$, which leads to the contradiction

$$d(x,v) = k + 1 \leq d(x,w) + d(w,v) = \ell + k - \ell - 1.$$

It remains the case when $w \notin P \cup Q$. If $P \cap R \neq \{x\}$, then $P \subset R$, otherwise we obtain a contradiction to the outerplanarity of G. But then $e\Theta g$ by Lemma 9. Thus $P \cap R = \{x\}$ and, similarly $Q \cap S = \{y\}$. If $P \cup R$ is not a shortest path (nor $Q \cup S$) we obtain a contradiction to outerplanarity again. Now we have $d(u,w) = d(v,z) = k + \ell$ and $d(u,z) = d(v,w) = k + \ell + 1$, which completes the proof. □

Since the property of being outerplanar is invariant under the removal of edges, we infer:

Proposition 11 *The class of bipartite outerplanar graphs is invariant under the removal of Θ-classes.*

A maximal connected subgraph without a cutvertex is called a *block*. Every block of a connected graph is either a maximal 2-connected subgraph or a bridge (with its ends), cf. [7].

We wish to remark that blocks of bipartite outerplanar graphs are edges or catacondensed benzenoid graphs. For more information on these graphs see [14] and references therein.

Proposition 12 *A bipartite outerplanar graph can be embedded into the hypercube in linear time.*

Proof Let G be a bipartite outerplanar graph. By Proposition 6 all faces of G can be recognized in linear time. We wish to determine the Θ-classes of G. Let G^+ be obtained as before by addition of an auxiliary vertex x.

First we separate all "tree-like" parts from G. In order to do that, we examine the faces of G^+ that contain an edge ux, where u is a vertex from G. We walk around the boundaries of these faces and mark all those edges of G that appear twice. Each such edge forms a Θ-class. We remove them from G together with the isolated vertices that may have appeared. Let G' be the resultant graph. Clearly G' has no "tree-like" parts and can be disconnected. Each component of G' is either a block or a component with a cut-vertex.

Let $\{H_1, H_2, ..., H_n\}$ be the set of blocks of G' and let e and f be edges of H_i and H_j, respectively, where $i \neq j$. Then, by Lemma 4, e and f cannot be in relation Θ. It thus suffices to find the Θ-classes of blocks H_i, $i = 1, ..., n$.

Let H_i be a block of G'. H_i is obviously outerplanar. First we observe (for instance by induction on the number of faces) that each face of H_i has a boundary that is an isometric cycle. The embedding is now effected by putting pairs of opposite edges of the faces of G' into the same Θ class.

By Theorem 10 this is a proper coloring. It is not hard to see that every edge is treated at most twice. Thus the procedure is linear. □

3 The algorithm

In this section we describe the algorithm that recognizes almost-median graphs with the property that all U_{ab}s are outerplanar. Denote this class of graphs by $AMOP$. Our result is a direct generalization of the algorithm in [4] that recognizes prism-free almost-median graphs, that is, almost-median graphs for which all $\langle U_{ab} \rangle$s are isometric trees. Our algorithm recognizes a larger class of graphs within the same time complexity. We will also show that there is no need for checking the isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ as in [4], since this is treated implicitly in other steps of the algorithm.

Note that the class $AMOP$ contains all prism-free almost-median graphs and, by Proposition 8, all planar almost-median graphs.

Let us first outline the algorithm. For the recognition of these graphs we first check whether they are bipartite and sparse, that is, we assume that G has at most $n \log n$ edges, cf. [13, Proosition 1.24]. Then we determine δ^*, U_{ab}^{δ}, U_{ba}^{δ}, and list the equivalence classes $E_1^{\delta}, E_2^{\delta}, \ldots, E_k^{\delta}$ of δ^* by listing all 4-cycles. Next we check whether the U_{ab}^{δ}s are outerplanar and embed them into the hypercube. This means that we determine the equivalence classes of U_{ab}^{δ} with respect to Θ. In the following steps we compare Θ and the δ^* classes on the U_{ab}. Finally, we show that $\Theta = \delta^*$, this ensures that Θ is transitive and the given graph a partial cube.

More precisely, we proceed as follows. Let $E_1^{\delta}, \ldots, E_k^{\delta}$ be the equivalence classes of δ^*, which we also call the *color classes with respect to* δ^*. We reconstruct G by joining these classes one by one, checking certain properties which necessarily have to hold if $\Theta = \delta^*$. If they are not satisfied, G cannot be semi-median (and thus not almost-median) and is rejected.

Algorithm A.

Input: A connected, bipartite graph G with $m \leq n \log n$.
Output: TRUE if G is almost-median, FALSE otherwise.

1. Determine δ^*, the U_{ab}^δ's, and the U_{ba}^δ's. Let $E_1^\delta, \ldots, E_k^\delta$ be the equivalence classes of δ^*.
2. Check whether every U_{ab}^δ is outerplanar. If not, return FALSE.
3. Embed every U_{ab}^δ in the hypercube i.e., properly color every U_{ab}^δ with respect to $\Theta(U_{ab}^\delta)$.
4. Check whether this coloring is compatible with δ^*. In other words, check whether edges that are in different $\Theta(U_{ab}^\delta)$ classes are in different δ^* classes. If not, return FALSE.
5. Let $G_0 = (V(G), \emptyset)$. For $i = 1, \ldots, n$ let $G_i = G_{i-1} \cup E_i^*$. If an edge of E_i^* has both endpoints in the same component of G_{i-1}, then return FALSE.
6. Return TRUE.

Lemma 13 *Every graph in the class AMOP is accepted by Algorithm A.*

Proof Let G be a graph in $AMOP$. Then G clearly passes Step 2. G is also almost-median and therefore a partial cube, that is, $\delta^* = \Theta^* = \Theta$. Since $U_{uv}^\delta = U_{uv}$, our graph G clearly passes Step 4. Concerning Step 5 we infer that if there exists an edge e of E_i^δ that has both endpoints in the same component of G_{i-1}, then e is in relation $\Theta = \delta^*$ to an edge of the component of G_{i-1} by Lemma 3.2 of [11]. Since this is not possible, G passes Step 5. $\square$

In the next lemma we are concerned with an isomorphism of the graphs $\left\langle U_{ab}^\delta \right\rangle$ and $\left\langle U_{ba}^\delta \right\rangle$. We have a bijection between the vertices of U_{ab}^δ and U_{ba}^δ that is induced by the ladder. The question is wether this bijection also induces a graph isomorphism between $\left\langle U_{ab}^\delta \right\rangle$ and $\left\langle U_{ba}^\delta \right\rangle$. This is important since $\left\langle U_{ab}^\delta \right\rangle$ and $\left\langle U_{ba}^\delta \right\rangle$ must be isomorphic for all partial cubes by [12, Lemma 3.2].

Lemma 14 *Let G be a graph and ab an edge in G for which $\left\langle U_{ab}^\delta \right\rangle$ and $\left\langle U_{ba}^\delta \right\rangle$ are not isomorphic. Then G is rejected by Algorithm A.*

Proof Suppose $\left\langle U_{ab}^\delta \right\rangle$ and $\left\langle U_{ba}^\delta \right\rangle$ are not isomorphic. Then U_{ab}^δ or U_{ba}^δ are not induced. We may assume without loss of generality that $e \in \left\langle U_{ab}^\delta \right\rangle$, $e \notin \left\langle U_{ba}^\delta \right\rangle$, and that the distance $d = d_{U_{ab}^\delta}(x, y)$ is minimal among all such edges $e = xy$. Denote this path of minimal length by P. If e is in a color class that is different from the color classes of the edges on P with respect to δ, then G is rejected in Step 5. So we can assume that $e \delta^* f$ for some $f \in P$. Let $x = x_0, x_1, \ldots, x_d = y$ be the vertices of P and

$x' = x'_0, x'_1, \ldots, x'_d = y'$ the vertices on the other side of the ladder. By minimality of d we have a ladder

$$x_0 x_d \delta^* x_1 x_{d-1} \delta^* \ldots x_{\lfloor \frac{d}{2} \rfloor} x_{\lceil \frac{d}{2} \rceil} \delta^* x'_{\lfloor \frac{d}{2} \rfloor} x'_{\lceil \frac{d}{2} \rceil} \delta^* \ldots \delta^* x'_1 x'_{d-1},$$

see Figure 2. But then the edges $x'_0 x'_1$ and $x'_{d-1} x'_d$ have different colors with respect to Θ, since $x'_0 x'_d$ is not an edge. On the other hand, they have the same color with respect to δ^* since

$$x'_0 x'_1 \delta^* x_0 x_1 \delta^* x_{d-1} x_d \delta^* x'_{d-1} x'_d$$

is a ladder. Thus G is rejected in Step 4.

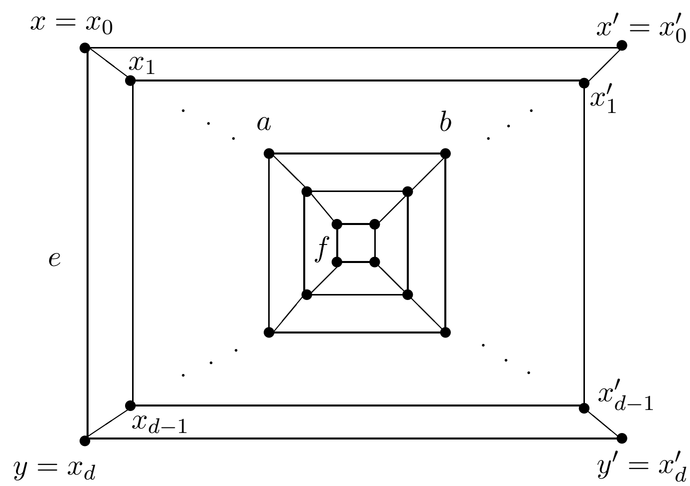

Figure 2: A situation from the proof of Lemma 14

□

Lemma 15 *Let G be a graph accepted by Algorithm A. Then $\delta^* = \Theta$.*

Proof Suppose this is not the case, then there are two edges that are either in relation Θ or in δ^*, but not in both. Let e, f be such a pair with minimum distance d in G. Let ℓ be the smallest index such that both e and f are in one and the same component of G_ℓ. This component will be denoted by $C_\ell^{e,f}$.

Claim 16 *Every shortest path from e to f is in $C_\ell^{e,f}$.*

Proof Let P be a shortest path from e to f. By Lemma 2 no two edges of P are in relation Θ. Hence, by the minimality of d we infer that they are also not in δ^*.

If P were not in G_ℓ, then it would contain edges in a class E_j^δ with $j > \ell$. Let j' be the largest such index. As all edges of P are in different classes E_j^δ, this implies that we add an edge to the graph $G_{j'-1}$ of which both endpoints are in one and the same component of $G_{j'-1}$, and so G is rejected in Step 5 of the algorithm. □

We continue with the proof of Lemma 15.

Case 1 $e\Theta f$.
In this case there are shortest paths P, Q of length d in G_ℓ between e and f such that $C = e \cup P \cup f \cup Q$ is a cycle. Note that neither e nor f can be in relation δ^* with any of the edges P or Q because of the minimality of d.

Assume first that no two edges of the cycle have the same δ^*-color. Then the algorithm rejects G in Step 5.

Thus suppose $g\delta^* h$, where $g = ab \in P$ and $h \in Q$. Clearly the distance between g and h is at most d. Let O be one side of the ladder between g and h. In O we can find the shortest path in U_{ab}^δ between corresponding end vertices of h and g. This can be done since every edge on this path is in different Θ classes of U_{ab}^δ by Step 3 and coloring in Step 3 is the same as coloring by Θ in G.

Denote this path by R. Let O' be the other side of the ladder, R' the isomorphic copy of R in O', S the part of C with the same endpoints as R and S' the one with the same endpoints as R', see Figure 3.

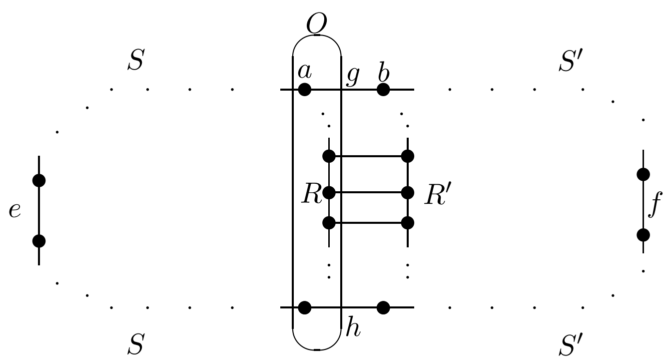

Figure 3: A ladder between g and h.

Again, without loss of generality, we may assume that $W = R \cup S$ is not longer than the walk $R' \cup S'$ and that W contains e. Then the length of W is at most $2d$. Clearly any two edges of W have distance at most $d - 1$. Thus, by Lemma 1 and the minimality of d, there is an edge on R that is in relation δ^* with e. By the minimality of d this edge is also in relation Θ to e.

Subcase 1.1 $|R| = d$.
Then S, R' and S' also have length d. The minimality of d implies that they all have

the same set of δ^*-colors. Hence e must be in relation δ^* to an edge of S'. Since this edge can be neither on P nor Q by the minimality of d it must be f.

Subcase 1.2 $|R| < d$.
Then $R' \cup g \cup S \cup h$ also has length at most $2d$. Hence there is an edge e' in R' with $e\delta^* e'$. Because of Step 5 we can find an edge e'' on S' with $e'\delta^* e''$. By the transitivity of δ^* we have $e'' = f$.

We have thus shown that e and f are in relation δ^* if they are in relation Θ.

Case 2 $e\delta^* f$.
By the Claim there is a shortest path P of length d between e and f, no two edges of which have the same δ^*-color. Consider a ladder between e and f. As before, it must be in G_ℓ and its sides must be shortest paths in G. But then $e\Theta f$. □

Theorem 17 *Algorithm A correctly recognizes almost-median graphs whose U_{ab}'s are outerplanar graphs and can be implemented to run in $O(m \log n)$ time.*

Proof By Lemma 13 almost-median graphs are accepted by Algorithm A. Suppose now that G is accepted. Then, by Lemma 15, $\Theta = \delta^*$. This implies that for any edge uv, $\left\langle U^\delta_{uv} \right\rangle = \langle U_{uv} \rangle$. Since the U^δ_{uv}'s are checked for outerplanarity in Step 2 and since U^δ_{uv} is isomorphic to $\left\langle U^\delta_{uv} \right\rangle$ by Lemma 14, G is an *AMOP* graph.

It remains to determine the complexity. For the determination of the quadrangles and squares of G we make use of the algorithm in [5] that has complexity $O(m\, a(G))$, where $a(G)$ denotes the *arboricity* of G, that is, the minimum number of disjoint spanning forests into which G can be decomposed. It considers every edge uv and all edges incident with an endpoint of uv of degree $\min\{d_G(u), d_G(v)\}$. Since

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} \leq 2a(G)m,$$

and since $a(G) \leq \log n$ for partial cubes, we can stop the algorithm if

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} > m \log n.$$

It follows that δ^* can be determined in $O(m \log n)$ time; cf. Proposition 7.6 (ii) of [13].

We can recognize outerplanar graphs and embed them into the hypercube in linear time by Propositions 6 and 12, respectively. Thus Step 2 and Step 3 remain within the required time complexity.

For Step 4 we observe that

$$\sum_{uv \in E(G)} |U^\delta_{uv}| = \sum_{j=1}^{k} |E^\delta_j| = m.$$

In Step 5 we have to perform two FIND SET operations and possibly one UNION for every one of the m edges in the graph. It is well known that these operations can be executed within time complexity $O(m \log n)$, cf. [6]. $\square$

4 Concluding remarks

It is not necessary that our graphs are planar if all U_{ab}'s are planar, Q_4 is an example of such a graph. Moreover, our graphs need not be planar even if all U_{ab}'s are outerplanar, as $K_{1,3}\square K_2$ shows. If G is planar, the time complexity becomes linear, because $a(G) \leq 4$ for planar graphs.

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